Complete extinction of solutions of a doubly nonlinear parabolic equation of fast diffusion type

Md Abu Hanif Sarkar", Masashi Misawa", and Kenta Nakamura†

"Department of Mathematics, Kumamoto University, Kumamoto 860-8555, Japan
"Faculty of Advanced Science and Technology, Kumamoto University, Kumamoto 860-8555, Japan
†Graduate School of Mathematics, Kyushu University, Fukuoka 819-0395, Japan

Abstract

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. For any positive $T \leq \infty$, let $\Omega_T := \Omega \times (0, T)$ be the space-time cylinder, and let $\partial_p \Omega_T$ be the parabolic boundary defined by $(\partial\Omega \times [0, T)) \cup (\Omega \times \{t = 0\})$. Throughout the paper we fix $p \in [2, n)$ and $q > p - 1$. We consider the following doubly nonlinear parabolic equation

$$\begin{cases} \partial_t(u^q) - \Delta_p u = 0 & \text{in} \quad \Omega_\infty \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega \end{cases} \tag{1}$$

Here the unknown function $u = u(x, t)$ is a nonnegative real-valued function defined for $(x, t) \in \Omega_\infty$, and the initial data u_0 is assumed to be in the Sobolev space $W_0^{1,p}(\Omega)$, positive, bounded in Ω and $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is the p-Laplacian.

In the case p = 2, the equation (1) becomes the so-called porous medium equation or the plasma equation. The global existence and continuity of a weak solution of (1) in the case p =2 is proved in ([2],[3], [4]). In particular, the complete extinction at a finite time of a continuous weak solution is shown in [2]. In the paper [1], studied the positivity of weak solutions in a space region for a fixed time for the plasma type equation.

We treat the doubly nonlinear equation (1) with p-Laplacian and study the positivity, boundedness and finite extinction of a weak solution of (1).
Our main assertion is the following :

Theorem 0.1. *Theorem (finite time complete extinction) Let u_0 be positive, bounded and in $W_0^{1,p}(\Omega)$. Let u be a nonnegative, continuous weak solution of (1). Then there exists a positive $T < \infty$ such that T is the complete extinction time for (1), that is, u is positive in $\Omega \times [0, T)$ and u vanishes in $\Omega \times [T, \infty)$.*

References

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