Weighted energy esitmates and applications for nonlinear wave equations

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Keel, Smith and Sogge have shown the estimate, so called “Keel-Smith-Sogge estimate,”

$$(\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}u'\|_{L^2((0,T)\times\mathbb{R}^3)} \lesssim \|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}+\int_0^T \|(\partial_t^2-\Delta)u(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds \quad (0.1)$$

in [11, Proposition 2.1], and apply it to the Cauchy problem for nonlinear wave equation

$$\begin{cases} (\partial_t^2-\Delta)u(t,x)=Q(u') & \text{for } (t,x)\in[0,\infty)\times\mathbb{R}^3 \\ u(0,x)=f(x),\quad \partial_t u(0,x)=g(x) & \text{for } x\in\mathbb{R}^3, \end{cases} \quad (0.2)$$

where Δ denotes the Laplacian, u' denotes the first derivatives of u by t and x, and $Q(u')$ denotes the quadratic term of u' such as $(\partial_t u)^2$ or $(\partial_t u)^2-|\nabla u|^2$. And they have shown the almost global solutions to (0.2) when the initial data are sufficiently small. Here, “almost global” solutions mean that the lifespan T of the solution u is as long as the exponential order, namely T can be taken as

$$T=C\exp\left(\frac{c}{\varepsilon}\right) \quad (0.3)$$

for some positive constants C and c, where

$$\varepsilon:=\sum_{|\alpha|\le 10}\|Z^\alpha f\|_{L^2(\mathbb{R}^3)}+\sum_{|\alpha|\le 9}\|Z^\alpha g\|_{L^2(\mathbb{R}^3)}, \quad (0.4)$$

and Z denotes the standard derivatives $\{\partial_t,\nabla_x\}$ and the angular derivatives $x_i\partial_j-x_j\partial_i$, $1\le i\ne j\le 3$. They have also considered the boundary-value problems for nonlinear wave equations. And now the estimate has become one of the useful estimates for the study of nonlinear wave equations.

Let us briefly review on the almost global solutions in the whole space $[0,\infty)\times\mathbb{R}^3$. It is known that the solution of (0.2) blows up in finite time for any nontrivial initial data when $Q(u')=(\partial_t u)^2$ (see [8], [23]), while the solution exists globally in time for sufficiently small initial data when $Q(u')=(\partial_t u)^2-|\nabla u|^2$ (see for example [25] and see [3], [14], [5, Theorem 6.6.2], [26, Theorem 5.1] for the null condition). The result for almost global solutions has already been known in [13]. And the lifespan $C\exp(c/\varepsilon)$ is known to be sharp in general for the blowing up solutions (see [9], [10], [21], [22] and also [23]). The estimate (0.1) gives a simpler proof for the almost global solutions, and (0.1) is proved in a simple way by the use of the classical energy estimates and the Huygens principle.

It is natural and useful to generalize the Keel-Smith-Sogge estimates to other dimensions and moreover to find other formulation. Metcalfe has constructed the Keel-Smith-Sogge estimates in more than 4 spatial dimensions in [16] by the method of harmonic analysis. In [27, Appendix (A.10)], Sterbenz and Rodnianski have given the alternative proof for Keel-Smith-Sogge estimates in more than 3 spatial dimensions based on the modified energy estimates without the use of the Huygens principle. And Metcalfe and Sogge have generalized them in [17]. In [6, Proposition B.2] (see also [7, Lemma 2.5 and its Remark]), Hidano and Yokoyama have considered the estimates in general dimensions based on the method in [16].

We should refer to the preceding results of the related weighted L^2 estimates for wave equations by Kenig, Ponce and Vega [12, Proposition 2.6], Mochizuki [18, §27], Matsuyama

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[15], Morawetz [19], [20], Smith and Sogge [24], and Strauss [28] (see also the description in [4, two lines after Lemma 2.2], [6, Remarks after Proposition B.1], [26, page 94], [1], [2]).

Since the vector fields method gives us the estimate

$$(1+t+|x|)^{(n-1)/2}(1+||t|-|x||)^{1/2}|u(t,x)| \lesssim \sum_{|\alpha|\leq(n+2)/2} \|\Gamma^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \tag{0.5}$$

for any function u on $(t-1,t+1)\times\mathbb{R}^n$ with suitable differentiability (see [5, p118,Proposition 6.5.1]), where Γ denotes the vector fields

$$\partial_t,\ \ \partial_j,\ \ t\partial_j+x_j\partial_t,\ \ x_j\partial_k-x_k\partial_j,\ \ 1\leq j\neq k\leq n,\qquad t\partial_t+\sum_{k=1}^n x_k\partial_k \tag{0.6}$$

and α denotes the multiple indices, we know by the classical energy estimates that the Cauchy problem

$$\begin{cases} (\partial_t^2-\Delta)u(t,x)=Q(u') & \text{for }\ (t,x)\in[0,\infty)\times\mathbb{R}^n \\ u(0,x)=f(x),\ \ \partial_t u(0,x)=g(x) & \text{for }\ x\in\mathbb{R}^n \end{cases} \tag{0.7}$$

has global solutions in cases $Q(u')=O(|u'|^2)$ with $n\geq 4$, $Q(u')=O(|u'|^3)$ with $n=3$, $Q(u')=O(|u'|^4)$ with $n=2$, and has almost global solutions in the cases $Q(u')=O(|u'|^2)$ with $n=3$, $Q(u')=O(|u'|^3)$ with $n=2$. While the vector fields method uses the operator $t\partial_j+x_j\partial_t$, the Keel-Smith-Sogge estimates use only $\partial_{t,x}$ and $x_j\partial_k-x_k\partial_j$ which are commutative with the c speed D'Alembertian $\partial_t^2-c^2\Delta$ which is not commutative with $t\partial_j+x_j\partial_t$ if $c\neq 1$. So that, the estimates are also useful when we consider multiple speed wave systems.

Let us first consider a generalization of Keel-Smith-Sogge estimates.

Lemma 0.1 *Let $n\geq 1$. Let $m>0$ be a real number. Let $\phi(\xi)=|\xi|^m$ for $\xi\in\mathbb{R}^n$. Then the operator $e^{\pm it\phi(i\nabla)}$ which is defined by*

$$e^{\pm it\phi(i\nabla)}=e^{\pm it\omega}=e^{\pm it(-\Delta)^{m/2}}:=\mathcal{F}^{-1}e^{\pm it\phi(\xi)}\mathcal{F}$$

satisfies the following estimates.

(1) If $m=1$, then

$$(\log(e+T))^{-1/2}\|\langle x\rangle^{-1/2}e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{0.8}$$

for any real number $T>0$.

(2) Let $h\in L^\infty((0,\infty),\mathbb{R})$ be a function which satisfies

$$H:=\left(\|h\|^2_{L^\infty(0<r<1)}+\sum_{j=0}^\infty \|h\|^2_{L^\infty(2^j<r<2^{j+1})}\right)^{1/2}<\infty.$$

Then

$$\|\langle x\rangle^{-1/2}h(|x|)e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)} \lesssim m^{-1/2}\|f\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)}. \tag{0.9}$$

We give some applications of the above lemma. We remember $\omega=(-\Delta)^{m/2}$.

Theorem 0.2 *Let $m\geq 1$ be any real number. Let $n\geq 2$, $M\geq n+m+1$, and $N:=M/2+n/2+1$. And let p be an integer which satisfies*

$$p\geq \max\left\{1+\frac{2}{n-1},2\right\}, \tag{0.10}$$

and let

$$\gamma := \frac{n-1}{4}\left(p-1-\frac{2}{n-1}\right). \tag{0.11}$$

Let λ be any real number. We consider the Cauchy problem

$$(P1) \quad \begin{cases} (\partial_t^2 + (-\Delta)^m)u(t,x) = \lambda\partial_x^{(m-1)/2+m}(u^p(t,x)) \quad \text{for } t \geq 0, \quad x \in \mathbb{R}^n \\ u(0,\cdot) = f(\cdot) \\ \partial_t u(0,\cdot) = g(\cdot), \end{cases} \tag{0.12}$$

where ∂_x^l denotes any derivative ∂_x^α with $|\alpha| = l$ if l is an integer, and the fractional derivative $\mathcal{F}^{-1}|\xi|^l\mathcal{F}$ if l is not an integer. We put

$$\varepsilon := \|\overline{Z}^M f\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M g\|_{\dot{H}^{-(m-1)/2-m}(\mathbb{R}^n)}. \tag{0.13}$$

If ε is sufficiently small, then the following results holds.

(1) If $p > 1 + 2/(n-1)$, then (P1) has a unique global solution in

$$\left\{ u \mid \|u\|_A := \|\langle x\rangle^{-1/2-\gamma}\overline{Z}^M u\|_{L^2((0,\infty)\times\mathbb{R}^n)} + \|\overline{Z}^N u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} < \infty \right\}. \tag{0.14}$$

(2) If $p = 1 + 2/(n-1)$ with $m = 1$, then (P1) has a unique almost global solution in

$$\begin{aligned} \Big\{ u \mid \|u\|_A := (\log(e+T))^{-1/2} & \|\langle x\rangle^{-1/2}\overline{Z}^M u\|_{L^2((0,T)\times\mathbb{R}^n))} \\ & + \|\overline{Z}^N u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))} < \infty \Big\}, \end{aligned} \tag{0.15}$$

where $T := C\exp(c/\varepsilon^{p-1})$ for some positive constants C and c.

(3) Let $T = \infty$ for (1). In (1) and (2), the solution u satisfies

$$Z^\alpha u \in C([0,T), L^2(\mathbb{R}^n)) \tag{0.16}$$

for any α with $|\alpha| \leq N$. And if $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ satisfy

$$\lim_{j\to\infty}\left\{\|\overline{Z}^M(f-f_j)\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M(g-g_j)\|_{\dot{H}^{-(m-1)/2-m}(\mathbb{R}^n)}\right\} = 0, \tag{0.17}$$

then the solution u_j for the Cauchy data f_j and g_j satisfies

$$\lim_{j\to\infty}\|u-u_j\|_A = 0. \tag{0.18}$$

Theorem 0.3 *Let $m \geq 1$ be any real number. Let $n \geq 2$, $M \geq n+m+3$, and $N := M-(m-1)/2$. And let p be an integer which satisfies*

$$p \geq \max\left\{1+\frac{2}{n-1}, 2\right\}, \qquad \text{and let} \qquad \gamma := \frac{n-1}{4}\left(p-1-\frac{2}{n-1}\right). \tag{0.19}$$

Let $P(\cdot,\cdot)$ be any homogeneous polynomial of p-order for two variables. We consider the Cauchy problem

$$(P2) \quad \begin{cases} (\partial_t^2 + (-\Delta)^m)u(t,x) = \partial_x^{(m-1)/2}(P(\partial_t u(t,x), \omega u(t,x))) \quad \text{for } t \geq 0, \quad x \in \mathbb{R}^n \\ u(0,\cdot) = f(\cdot) \\ \partial_t u(0,\cdot) = g(\cdot), \end{cases} \tag{0.20}$$

where ∂_x^l denotes any derivative ∂_x^α with $|\alpha| = l$ if l is an integer, and the fractional derivative $\mathcal{F}^{-1}|\xi|^l\mathcal{F}$ if l is not an integer. We put

$$\varepsilon := \|\overline{Z}^M f\|_{H^{(m+1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M g\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}. \tag{0.21}$$

If ε is sufficiently small, then the following results hold.

(1) If $p > 1 + 2/(n-1)$, then (P2) has a unique global solution in

$$\begin{aligned}\{\ \ u \mid \|u\|_A := \|\langle x\rangle^{-1/2-\gamma}\overline{Z}^M\partial_t u\|_{L^2((0,\infty)\times\mathbb{R}^n)} + \|\langle x\rangle^{-1/2-\gamma}\overline{Z}^M\omega u\|_{L^2((0,\infty)\times\mathbb{R}^n)}\\ + \|\overline{Z}^N\partial_t u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} + \|\overline{Z}^N\omega u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))}\\ + \sup_{0<t<\infty}(1+t)^{-1}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty\ \}\,. \end{aligned} \tag{0.22}$$

(2) If $p = 1 + 2/(n-1)$ with $m = 1$, then (P2) has a unique almost global solution in

$$\begin{aligned}\{\ \ u \mid \|u\|_A := (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\overline{Z}^M\partial_t u\|_{L^2((0,T)\times\mathbb{R}^n)}\\ + (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\overline{Z}^M\omega u\|_{L^2((0,T)\times\mathbb{R}^n)}\\ + \|\overline{Z}^N\partial_t u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))} + \|\overline{Z}^N\omega u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))}\\ + \sup_{0<t<T}(1+t)^{-1}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty\ \}\,, \end{aligned} \tag{0.23}$$

where $T := C\exp(c/\varepsilon^{p-1})$ for some positive constants C and c.

(3) Let $T = \infty$ for (1). In (1) and (2), the solution u satisfies

$$Z^\alpha u \in C([0,T), L^2(\mathbb{R}^n)) \tag{0.24}$$

for any α with $|\alpha| \le N$. And if $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ satisfy

$$\begin{aligned}\lim_{j\to\infty}\Big\{\|\overline{Z}^M(f-f_j)\|_{H^{(m+1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M(g-g_j)\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)}\\ + \|g-g_j\|_{L^2(\mathbb{R}^n)}\Big\}\,, \end{aligned} \tag{0.25}$$

then the solution u_j for the Cauchy data f_j and g_j satisfies

$$\lim_{j\to\infty}\|u-u_j\|_A = 0. \tag{0.26}$$

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