On fundamental solutions to fractional diffusion equations with divergence free drift

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This talk is based on a joint work with Hideyuki Miura (Osaka university). We are interested in the drift diffusion equations with the fractional Laplacian:

$$\partial_t\theta + (-\Delta)^{\frac{\alpha}{2}}\theta + v\cdot\nabla\theta = 0, \quad \mathrm{div}\ v = 0, \qquad t>0, \quad x\in\mathbb{R}^d, \tag{1}$$

where $d\geq 2$ and $(-\Delta)^{\alpha/2}$ is formally defined by

$$\big(-\Delta\big)^{\frac{\alpha}{2}} f(x) = P.V.\int_{\mathbb{R}^d}\frac{f(x)-f(y)}{|x-y|^{d+\alpha}}dy. \tag{2}$$

Here $\alpha\in(0,2)$ is a constant and $v=v(t,x)$ is a given solenoidal vector field in $\mathbb{R}^d$ with suitable regularity to be specified later. The equations of the form (1.1) appear in several models in the fluid dynamics, where the solution describes some active scalar convected by the incompressible flows.

In the case $d=2$, if v and θ are related by $v=(R_2\theta,-R_1\theta)$, the equation (1) is called the dissipative quasi-geostrophic equation (QG) [4]. Here R_i $(i=1,2)$ is the Riesz transform. In particular, the case $\alpha=1$ is called critical in the sense that both the diffusion term and the drift term become the leading terms of the equation. The global regularity of the solution in the critical case is recently addressed in [7, 2, 6]. Due to the non-local character of the fractional Laplacian, the usual local regularity theory does not work and the nonsmooth drift term makes problems more delicate. Indeed, by maximum principle and the boundedness of the Riesz transform R_i, it can be shown that v belongs to BMO, However the critical case is the boaderline where the bootstrap argument can be performed to obtain the regularity of the solution in view of the scaling.

In [2, 6] they regarded (QG) as the diffusion equation with a given drift term in BMO. Motivated by these results, we study the existence and the regularity of fundamental solutions of (1) under weak assumptions for v. Let $1<q<\infty$ and set

$$X_\alpha = \begin{cases} L^\infty(0,\infty;L^{\frac{d}{\alpha-1}}(\mathbb{R}^d)) & \text{if } \alpha\in(1,2),\\ L^\infty(0,\infty;BMO(\mathbb{R}^d)) \ \cap\ L^q_{loc}(0,\infty;L^1_{loc}(\mathbb{R}^d)) & \text{if } \alpha=1,\\ L^\infty(0,\infty;\dot{C}^{1-\alpha}(\mathbb{R}^d)) \ \cap\ L^q_{loc}(0,\infty;L^\infty_{loc}(\mathbb{R}^d)) & \text{if } \alpha\in(0,1),\end{cases}$$

$$Y = L^1(0,\infty;\mathrm{Lip}(\mathbb{R}^d)) \ \cap\ L^q_{loc}(0,\infty;L^\infty_{loc}(\mathbb{R}^d)).$$

Theorem. *Let $\alpha\in(0,2)$. Suppose that v belongs to either X_α^d or Y^d. Then there exists a fundamental solution $P_{\alpha,v}(t,x;s,y)$ for (1) satisfying the following properties.*

$$\int_{\mathbb{R}^d} P_{\alpha,v}(t,x;s,y)dx = \int_{\mathbb{R}^d} P_{\alpha,v}(t,x;s,y)dy = 1, \tag{3}$$

$$P_{\alpha,v}(t,x;s,y) = \int_{\mathbb{R}^d} P_{\alpha,v}(t,x;\tau,z)P_{\alpha,v}(\tau,z;s,y)dz, \tag{4}$$

$$|P_{\alpha,v}(t,x_1;s,y_1) - P_{\alpha,v}(t,x_2;s,y_2)| \leq \frac{C\big(|x_1-x_2|^\beta + |y_1-y_2|^\beta\big)}{(t-s)^c}, \tag{5}$$

and for $T \geq t_i > s_i \geq 0$, $i = 1, 2$,

$$|P_{\alpha,v}(t_1, x; s_1, y) - P_{\alpha,v}(t_2, x; s_2, y)| \leq \frac{C_{T,x}|t_1 - t_2|^{\beta'} + C_{T,y}|s_1 - s_2|^{\beta'}}{(\min\{t_1 - s_1, t_2 - s_2\})^{c'}}. \tag{6}$$

Moreover, if $v \in X^d_\alpha$ then

$$P_{\alpha,v}(t, x; s, y) \leq C(t-s)^{-\frac{d}{\alpha}} \left(1 + \frac{\big(|x-y| - 2M[v](t, s, x, y)\big)_+}{(t-s)^{\frac{1}{\alpha}}} \right)^{-d-\alpha}, \tag{7}$$

$$M[v](t, s, x, y) = \sup_{s<r<t} \Big| \int_s^r \mathrm{Avg}_{B_{|x-y|}(x)} v(\tau) d\tau \Big|. \tag{8}$$

Even in the case $v \in Y^d$, (7) holds if $|t - s| \ll 1$ or $\|v\|_{L^1(0,\infty;\mathrm{Lip}(\mathbb{R}^d))} \ll 1$. The positive constants C, c, c', β, β' are uniform in time and space, and the positive constant $C_{T,x}$ (or $C_{T,y}$) satisfies $\sup_{|x|\leq R} C_{T,x} < \infty$ (or $\sup_{|y|\leq R} C_{T,y} < \infty$) for each $R > 0$.

Remarks (i) When $\alpha \in (1, 2)$ and v belongs to a suitable Kato class without the condition div $v = 0$, the fundamental solutions of (1) were constructed in [1, 5]. In this case the diffusion term is the leading term and then perturbation arguments work. Under the divergence free condition, our result relaxes the restriction of α and the regularity assumptions for v.
(ii) In order to construct fundamental solutions, it is essential to show the a priori bounds related with the estimates in the theorem. To this end we will develop the Nash-type arguments in [3, 8, 9] which studied the non-local diffusion equations without drift terms. As in [10, 8, 9], the argument to derive the continuity estimates consists of four steps; the moment bound, the relative entropy bound, the overlap estimate, and the iteration estimate. However, due to the presence of the nonsmooth drift term, it is not easy to get these estimates. Motivated by [2, 6], we estimate fundamental solutions in time-dependent coordinates along the trajectory determined by a local average of v. Our method can be adapted for more general non-local diffusion equations associated with certain Dirichlet forms as in [8, 9].
(iii) The regularity assumption for v is invariant under the scaling $v(t, x) \mapsto \lambda^{\alpha-1} v(\lambda^\alpha t, \lambda x)$, which is compatible with the natural scaling of the equation (1).

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