Level statistics for a 1-dim random Schroedinger operator

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We consider the following operator.

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on } L^2(\mathbf{R})$$

$a \in C^\infty(\mathbf{R})$, $a(-t) = a(t)$, a is non-increasing for $t > 0$. We assume

$$ct^{-\alpha} \le a(t) \le Ct^{-\alpha}, \quad t \gg 1$$

for some $c, C > 0$, $\alpha > 0$, which we denote by $a \sim t^{-\alpha}$. F is a non-constant smooth function on a Riemanian manifold, and X_t is the Brownian motion on M. It is known that[3]
(1) $\alpha < \frac{1}{2}$: $\sigma(H) \cap [0, \infty)$ is pure point
(2) $\alpha = \frac{1}{2}$: we have $E_0 > 0$ s.t. $[0, E_0]$ is pure point and $[E_0, \infty)$ is singular continuous
(3) $\alpha > \frac{1}{2}$: $\sigma(H) \cap [0, \infty)$ is absolutely continuous.
Let $H_L := H|_{[0,L]}$ with Dirichlet bc and let $\{E_n(L)\}_{n \ge 1}$ be the nonnegative eigenvalues of H_L arranged in increasing order. We take arbitrary $E_0 > 0$ as the reference energy and consider the following point process.

$$\xi_L := \sum_n \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0(L)})}.$$

Definition
Let μ be a probability measure on $[0, \pi)$. We say the point process ξ is the clock process with spacing π associated to μ if and only if[1]

$$\mathbf{E}[e^{-\xi(f)}] = \int_0^\pi d\mu(\phi) \exp\left(-\sum_{n \in \mathbf{Z}} f(n\pi + \phi) \right), \quad f \in C_c(\mathbf{R}).$$

(A)
Fix $\beta \in [0, \pi)$. A subsequence $\{L_k\}_{k \ge 1}$ satisfies

$$\sqrt{E_0} L_k = k\pi + \beta + o(1), \quad k \to \infty$$

[1] $\xi(f) := \int_{\mathbf{R}} f(x)\xi(dx)$

Theorem 1(Convergence to a clock process)
Assume (A) and $\alpha > \frac{1}{2}$. Then we can find a probability measure μ on $[0, \pi)$ s.t. ξ_{L_k} converges in distribution to the clock process with spacing π associated to μ.

Theorem 1 has been proved for CMV matrices with μ the uniform distribution[2]. Next we rearrange $\{E_n(L)\}$ so that

$$\cdots \tilde{E}_{-1}(L) < E_0 \leq \tilde{E}_0(L) < \cdots$$

Theorem 2(Stong clock behavior)
For any $j \in \mathbf{Z}$, $\lim_{L\to\infty} L(\sqrt{E_{j+1}(L)} - \sqrt{E_j(L)}) = \pi$, a.s.

Theorem 2 has been proved for some classes of Jacobi matrices[1]. Proof of Theorems follow from the analysis done in [3, 4].

References

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