A DYNAMICAL APPROACH TO THE LARGE TIME ASYMPTOTICS FOR WEAKLY COUPLED SYSTEMS OF HAMILTON–JACOBI EQUATIONS

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I will talk on the study which is based on the recent joint work [3] with Hung V. Tran from the University of Chicago. In this talk, we will discuss on the large-time behavior of the value functions of the optimal control problems on the n-dimensional torus which appear in the dynamic programming for the system whose states are governed by random changes. More precisely, we consider the minimizing problem:

$$\text{Minimize} \quad \mathbb{E}_i\Big[\int_{-t}^{0} L_{\nu(s)}(\gamma(s),\dot\gamma(s))\,ds + g_{\nu(-t)}(\gamma(-t))\Big], \tag{0.1}$$

over all controls $\gamma\in \mathrm{AC}\,([-t,0])$ with $\gamma(0)=x$ for any fixed $(x,t)\in\mathbb{T}^n\times[0,\infty)$, where the Lagrangians $L_i(x,v):\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{R}$ are derived from the Fenchel-Legendre transforms of given Hamiltonians H_i and we denote by $\mathrm{AC}\,([-t,0])$ the set of absolutely continuous functions on $[-t,0]$ with values in $\mathbb{T}^n$. The functions g_i are given real-valued continuous functions on $\mathbb{T}^n$ for $i=1,2$. Here $\mathbb{E}_i$ denotes the expectation of a process with $\nu(0)=i$, where ν is a $\{1,2\}$-valued process which is a continuous-time Markov chain on $(-\infty,0]$ (notice that time is reversed) such that for $s\le 0$, $\Delta s>0$,

$$\mathbb{P}\big(\nu(s-\Delta s)=j \mid \nu(s)=i\big) = c_i\Delta s + o(\Delta s) \text{ as } \Delta s\to 0 \text{ for } i\ne j, \tag{0.2}$$

where c_i are given positive constants and $o:[0,\infty)\to[0,\infty)$ is a function satisfying $o(r)/r\to 0$ as $r\to 0$.

From the point of view of the study on partial differential equations, it is equivalent to consider viscosity solutions of quasi-monotone weakly coupled systems of Hamilton–Jacobi equations

$$\text{(C)}\qquad \begin{cases} (u_1)_t + H_1(x,Du_1) + c_1(u_1-u_2) = 0 & \text{in } \mathbb{T}^n\times(0,\infty),\\ (u_2)_t + H_2(x,Du_2) + c_2(u_2-u_1) = 0 & \text{in } \mathbb{T}^n\times(0,\infty),\\ u_i(x,0)=g_i(x) & \text{on } \mathbb{T}^n, \end{cases}$$

where the Hamiltonians $H_i(x,p):\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{R}$ are given continuous functions for $i=1,2$, which are assumed to satisfy the followings:

(A1) The functions H_i are uniformly coercive in the p-variable, i.e.,

$$\lim_{r\to\infty}\inf\{H_i(x,p)\mid x\in\mathbb{T}^n, |p|\ge r\}=\infty.$$

(A2) The functions $p\mapsto H_i(x,p)$ are strictly convex for any $x\in\mathbb{T}^n$.

Here u_i are real-valued unknown functions on $\mathbb{T}^n\times[0,\infty)$ and $(u_i)_t=\partial u_i/\partial t$, $Du_i=(\partial u_i/\partial x_1,\dots,\partial u_i/\partial x_n)$ for $i=1,2$, respectively.

The large-time behavior of viscosity solutions of this problem has been recently started to study by the authors in [2] and Camilli, Ley, Loreti, and Nguyen in [1] for some special cases, independently. However, the general cases remain widely open and the techniques developed in [2, 1] are not applicable for general cases. The coupling terms cause serious difficulties, which will be explained in the talk.

We will explain the dynamical approach on this problem. The key ingredients in this approach consist of obtaining existence and stability results of extremal curves of (0.1). It is fairly straightforward to prove the existence of extremal curves by using techniques from calculus of variations. However, representation formulas (0.1) are implicit in some sense and prevent us from deriving a stability result. In order to overcome this difficulty, we give more deterministic formulas for the value functions of (0.1) by explicit calculations. By using the new formulas, which are more intuitive, we are able to derive the large time behavior results.

References

1. F. Camilli, O. Ley, P. Loreti and V. Nguyen, *Large time behavior of weakly coupled systems of first-order Hamilton–Jacobi equations*, to appear in NoDEA.
2. H. Mitake, H. V. Tran, *Remarks on the large-time behavior of viscosity solutions of quasi-monotone weakly coupled systems of Hamilton–Jacobi equations*, Asymptot. Anal., **77** (2012), 43–70.
3. H. Mitake, H. V. Tran, A dynamical approach to the large-time behavior of solutions to weakly coupled systems of Hamilton–Jacobi equations, submitted.

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