# A quasilinear parabolic perturbation of the linear heat equation[1]

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This talk is based on joint works with S. Cano-Casanova and J. López-Gómez [2, 3].

We deal with the dynamics of the parabolic quasilinear boundary value problem

$$\begin{cases} \partial_t u - \left(\frac{u'}{\sqrt{1+\kappa(u')^2}}\right)' = \lambda u, & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(\cdot, 0) = u_0 > 0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$, $\kappa > 0$, $u_0 \in C[0,1]$, and $'$ stands for the spatial derivative

$$' = D := \frac{\partial}{\partial x}.$$

This problem establishes a *quasilinear continuum deformation* between the *linear parabolic problem*

$$\begin{cases} \partial_t u - D^2 u = \lambda u, & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(\cdot, 0) = u_0 > 0, \end{cases} \tag{2}$$

and the problem (1), which is a parabolic problem associated to the one-dimensional mean curvature operator. It is well known that the unique solution to (2) is given from the linear hear semigroup through

$$u(x,t;u_0) = e^{(D^2+\lambda)t} u_0$$

and, consequently,

$$\lim_{t\uparrow\infty} u(x,t;u_0) = \begin{cases} 0, & \text{if } \lambda < \pi^2, \\ \infty, & \text{if } \lambda > \pi^2, \end{cases} \quad \text{for all } x \in (0,1),$$

whereas at $\lambda = \pi^2$, the problem (2) possesses a straight half-line of positive steady-states, namely all positive multiples of $\sin(\pi x)$.

Although the non-negative steady-states of (2) are given through the linear eigenvalue problem

$$\begin{cases} -u'' = \lambda u, & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

the steady-states of (1) are given by the non-negative solutions to the quasilinear boundary value problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+\kappa(u')^2}}\right)' = \lambda u, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

The main results of this talk concerning the existence of positive solutions for (3) can be summarized in the following list:

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- Problem (3) has a positive solution if and only if

$$8B^2 < \lambda < \pi^2, \qquad B := \int_0^1 \frac{d\theta}{\sqrt{\theta^{-4}-1}}. \tag{4}$$

- The positive solution to (3) is unique if it exists. Subsequently we denote it by $u_\lambda$.
- $u_\lambda$ is symmetric around 1/2 for all $\lambda$ satisfying (4).
- $u_\lambda$ satisfies

$$\lim_{\lambda\downarrow 8B^2} u_\lambda'(0)=\infty, \quad \lim_{\lambda\downarrow 8B^2} \|u_\lambda\|_\infty = \frac{1}{2B\sqrt{\kappa}} \text{ and } \lim_{\lambda\uparrow \pi^2} \|u_\lambda\|_\infty = 0. \tag{5}$$

The bifurcation diagram can be written from the results above. Note that the interval of values of $\lambda$ for which (3) admits a positive solution does not depend on the value of $\kappa > 0$, though $\kappa$ measures the maximal size of all positive steady-states $u_\lambda$ through (5).

As we are dealing with *a potential superlinear at infinity*, due to Bonheure et al. [1], there exists $\lambda_c > 0$ such that for every $\lambda \in (0, \lambda_c)$, (3) has a weak solution. On the other hand, according to the results stated above, (3) cannot admit a positive classical solution for small $\lambda > 0$. Therefore, the weak solutions of Bonheure et al. in [1] must be *nonclassical* for small $\lambda > 0$, i.e., they are regular in $(0,1)$ but present a derivative blow-up at 0 and 1. These weak solutions seem to be *bounded variation solutions* as discussed by Obersnel and Omari [6], and, hence [6] provides us with a multidimensional counterpart of [1].

Also recently, Mellet and Vovelle [5] have proven that if $H > 0$, then any weak solution to the perturbed problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+\kappa(u')^2}}\right)' = H + \lambda f(u), & 0 < x < 1 \\ u(0)=0,\ u(1)=0, \end{cases}$$

must be classical. Consequently, in general, there is a huge difference between the general case when $H > 0$ and the special case when $H = 0$, where the solutions cannot be classical for sufficiently small $\lambda > 0$. But, rather naturally, the techniques from Mellet and Vovelle [5] can be adapted to prove that any solution to (3) in the class $\mathrm{Lip}(0,1)$ must be a classical solution. As a byproduct, weak solutions of Bonheure et al. [1] for sufficiently small $\lambda > 0$ cannot lie in the class $\mathrm{Lip}(0,1)$, whence $|u'(0)| = |u'(1)| = \infty$.

As far as concerns to the dynamics of (1), the main findings, in the most interesting case when the condition (4) holds, can be shortly summarized as follows:

- For every $\lambda \in (8B^2, \pi^2)$, $u_\lambda$ is linearly unstable, as a steady-state of (1). Moreover,
  (a) If $u_0 < \eta u_\lambda$ for some $0 < \eta < 1$, then

  $$\lim_{t\uparrow\infty} u(x,t;u_0) = 0 \quad \text{uniformly in } [0,1],$$

  where $u(x,t;u_0)$ stands for the unique solution to (1).

  (b) If $u_0 > \eta u_\lambda$ for some $\eta > 1$, then

  $$\lim_{t\uparrow\infty} u(x,t;u_0) = \infty \quad \text{uniformly in every compact subset of } (0,1).$$

Considering also the other cases, we can get a rather complete panorama of the dynamics of (1). We should also remark here that no matter how small the parameter $\lambda > 0$ is, the solutions to (1) can be grown as much as we want by choosing the initial values $u_0$ sufficiently large, as observed numerically by Marcellini and Miller [4].

In this talk, we also consider the following parabolic quasilinear boundary value problem

$$\begin{cases} \partial_t u - \left( \frac{u'}{\sqrt{1+\kappa(u')^2}} \right)' = \lambda V(x) u, & 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(\cdot, 0) = u_0 > 0. \end{cases} \tag{6}$$

We note that the special case when $V = 1$ reduces to (1). The results for the problem (6) will be presented in the talk. As a first difficulty, to study the classical steady-states of (6), which are the non-negative classical solutions to the quasilinear elliptic problem

$$\begin{cases} -\left( \frac{u'}{\sqrt{1+\kappa(u')^2}} \right)' = \lambda V(x) u, & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{7}$$

one cannot use phase portrait technique, as is done in the case $V = 1$. As a consequence of this handicap, it remains an open problem to ascertain whether (7) admits a unique positive classical solution or not.

## References


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