LOGARITHMIC TIME DECAY FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATIONS

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1. Introduction

We study the Cauchy problem for the one dimensional cubic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = \mathcal{N},\ x \in \mathbf{R},\ t > 1, \\ u(1,x) = u_0(x),\ x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $\mathcal{N} = \lambda e^{i\frac{\pi}{2}} u^3 + |u|^2 u$, $0 < |\lambda| < \sqrt{3}$. A more general cubic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = \lambda e^{i\omega} u^3 + \kappa |u|^2 u,\ x \in \mathbf{R},\ t > 1, \\ u(1,x) = u_0(x),\ x \in \mathbf{R}, \end{cases}$$

where $\lambda, \omega, \kappa \in \mathbf{R}$, $\lambda, \kappa \neq 0$, can be reduced to (1.1) by changing the dependent variable $u = \frac{1}{\kappa} e^{i\frac{\gamma}{2} - i\frac{\pi}{4}} v$, and replacing $\frac{\lambda}{\kappa}$ by λ.

2. Previous Works

In the past works, it was studied the Cauchy problem for the cubic derivative nonlinear Schrödinger equation

$$iu_t + \frac{1}{2}u_{xx} = \mathcal{N}$$

with a nonlinearity $\mathcal{N}$ decomposed in two parts $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$, where

$$\begin{aligned} \mathcal{N}_1 &= \lambda_1 |u|^2 u + i\lambda_2 |u|^2 u_x + i\lambda_3 u^2 \overline{u}_x \\ &\quad + \lambda_4 |u_x|^2 u + \lambda_5 \overline{u} u_x^2 + i\lambda_6 |u_x|^2 u_x \end{aligned}$$

satisfies a gauge invariance property $\mathcal{N}_1(e^{i\theta}u) = e^{i\theta}\mathcal{N}_1(u)$ for all $\theta \in \mathbf{R}$ and

$$\begin{aligned} \mathcal{N}_2 &= \lambda_7 u^2 u_x + \lambda_8 u u_x^2 + \lambda_9 u_x^3 \\ &\quad + \lambda_{10} \overline{u}^2 \overline{u}_x + \lambda_{11} \overline{u}\overline{u}_x^2 + \lambda_{12} \overline{u}_x^3 \\ &\quad + \lambda_{13} \overline{u}^2 u_x + \lambda_{14} |u|^2 \overline{u}_x \\ &\quad + \lambda_{15} u \overline{u}_x^2 + \lambda_{16} \overline{u} |u_x|^2 + \lambda_{17} |u_x|^2 \overline{u}_x \end{aligned}$$

is not gauge invariant, the coefficients

$$\lambda_1, \lambda_6 \in \mathbf{R}, \lambda_j \in \mathbf{C}, 2 \leq j \leq 5, 7 \leq j \leq 17$$

are such that

$$\lambda_2 - \lambda_3, \lambda_4 - \lambda_5 \in \mathbf{R}.$$

Key words and phrases. This is a joint work with P.I.Naumkin.

It was shown that if the initial data

$$u_0 \in \mathbf{H}^3 \cap \mathbf{H}^{2,1}$$

have a sufficiently small norm

$$\|u_0\|_{\mathbf{H}^3} + \|u_0\|_{\mathbf{H}^{2,1}},$$

then there exists a unique solution

$$u \in \mathbf{C}\left([1,\infty);\mathbf{H}^3 \cap \mathbf{H}^{2,1}\right),$$

where the weighted Sobolev space is

$$\mathbf{H}^{m,k} = \left\{\phi \in \mathbf{S}' : \|\phi\|_{\mathbf{H}^{m,k}} = \left\| \langle x \rangle^k \langle i\partial \rangle^m \phi \right\|_{\mathbf{L}^2} < \infty \right\}$$

with $m, k \in \mathbf{R}$, $\langle x \rangle = \sqrt{1+x^2}$. We denote $\mathbf{H}^m = \mathbf{H}^{m,0}$. Moreover it was obtained that there exists a unique final state $W_+ \in \mathbf{L}^\infty$ such that the following asymptotics is valid

$$\begin{aligned} u(t,x) &= (it)^{-\frac{1}{2}} e^{\frac{ix^2}{2t}} W_+\left(\frac{x}{t}\right) \\ &\quad \times \exp\left(i\Lambda\left(\frac{x}{t}\right)\left|W_+\left(\frac{x}{t}\right)\right|^2 \log t\right) \\ &\quad + O\left(t^{-\frac{1}{2}-\nu}\right) \end{aligned}$$

for large time t uniformly with respect to $x \in \mathbf{R}$, where

$$\Lambda(\xi) = \lambda_1 - (\lambda_2 - \lambda_3)\xi + (\lambda_4 - \lambda_5)\xi^2 - \lambda_6 \xi^3$$

and $\nu \in \left(0, \frac{1}{4}\right)$. Here the logarithmic oscillation is due to a gauge invariant part $\mathcal{N}_1$, whereas $\mathcal{N}_2$ does not affect explicitly the asymptotic behavior.

The non gauge invariant cubic nonlinearities without derivatives of the unknown function are more difficult to study. We first survey the final value problems. In a work by [6], existence of the wave operator for

$$iu_t + \frac{1}{2}u_{xx} = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3 \tag{2.1}$$

was shown, where $\lambda_j \in \mathbf{C}$, $j = 1,2,3$. More precisely, if the final state

$$u_+ \in \mathbf{H}^{0,3} \cap \dot{\mathbf{H}}^{-4}$$

and the norm

$$\|u_+\|_{\mathbf{H}^{0,3}} + \|u_+\|_{\dot{\mathbf{H}}^{-4}}$$

is sufficiently small, then there exists a unique global solution $u \in \mathbf{C}\left([1,\infty);\mathbf{L}^2\right)$ of (2.1) such that

$$\lim_{t\to\infty} \left\| u(t) - e^{\frac{it}{2}\partial_x^2} u_+ \right\|_{\mathbf{L}^2} = 0, \tag{2.2}$$

where the homogeneous Sobolev space is

$$\dot{\mathbf{H}}^m = \left\{ u \in \mathcal{S}'; \|u\|_{\dot{\mathbf{H}}^m} = \left\| \left(-\partial_x^2\right)^{\frac{m}{2}} u \right\|_{\mathbf{L}^2} < \infty \right\}.$$

This result was improved by [5] after the works by [7] and [4] as follows. If the final state

$$u_+ \in \mathbf{H}^{0,\alpha} \cap \dot{\mathbf{H}}^{-\alpha}$$

with $\alpha > \frac{1}{2}$ and the norm

$$\|u_+\|_{\mathbf{H}^{0,\alpha}} + \|u_+\|_{\dot{\mathbf{H}}^{-\alpha}}$$

is sufficiently small, then there exists a unique global solution $u \in \mathbf{C}\left([1,\infty);\mathbf{L}^2\right)$ of (2.1) such that (2.2) holds. Note that the vanishing condition at the origin for the final data $\widehat{u}_+(0) = 0$ was assumed in these works. Here the Fourier transform is defined by

$$\mathcal{F}\phi = \widehat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x)\, dx.$$

It seems that if the vanishing condition for the final data is not satisfied, then the asymptotic behavior of solutions of (2.1) is different from that of the free solution.

We note that if $\phi \in \mathbf{H}^{0,2}$ and $\widehat{\phi}(0) = 0$, then $\phi \in \mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{-\alpha}$ for $0 \leq \alpha < 1 + \frac{n}{2}$ with $n = 1, 2$.

Global existence and asymptotic behavior of solutions to the Cauchy problem for (2.1)

$$iu_t + \frac{1}{2}u_{xx} = \lambda_1 u^3$$

was studied by [3] and global existence of small solutions was shown in the case of the initial data $u_0 \in \mathbf{H}^{\alpha} \cap \mathbf{H}^{0,\alpha}$ with $\alpha > \frac{1}{2}$. More precisely, if $u_0 \in \mathbf{H}^{\alpha} \cap \mathbf{H}^{0,\alpha}$ with $\alpha > \frac{1}{2}$ and the argument and amplitude conditions

$$\sup_{|\xi|\leq 1} \left|\arg \widehat{u_0}(\xi)\right| < \frac{\pi}{8}, \quad \inf_{|\xi|\leq 1} \left|\widehat{u_0}(\xi)\right| \geq \varepsilon^{\frac{5}{4}}$$

are satisfied, then there exists a unique solution $u \in \mathbf{C}\left([1,\infty);\mathbf{H}^{\alpha} \cap \mathbf{H}^{0,\alpha}\right)$ of the Cauchy problem of the above problem. Moreover the asymptotics

$$u(t,x) = \frac{(it)^{-\frac{1}{2}} \left|\widehat{u_0}\left(\frac{x}{t}\right)\right| \exp\left(\frac{ix^2}{2t}\right)}{\left(1 + \frac{\left|\widehat{u_0}\left(\frac{x}{t}\right)\right|^2}{\sqrt{3}} \log \frac{t}{1+\frac{x^2}{t}}\right)^{\frac{1}{2}}} + O\left(t^{-\frac{1}{2}} \left(\log \frac{t}{1+\frac{x^2}{t}}\right)^{-\frac{1}{2}-\nu}\right) \tag{2.3}$$

is valid as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\nu > 0$ is small. Asymptotics (2.3) implies that the following time decay estimate for solutions is fulfilled

$$\sup_{|x|\leq\sqrt{t}} |u(t,x)| \leq C\varepsilon t^{-\frac{1}{2}} \left(1 + \varepsilon^2 \log t\right)^{-\frac{1}{2}}.$$

Thus solutions obtain a more rapid time decay in the short-range region $|x| \leq \sqrt{t}$ comparing with the linear case. If we assume a more strong condition such that the data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$ are odd functions, then it was shown by [2] that solutions of

$$iu_t + \frac{1}{2}u_{xx} = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3$$

are asymptotically free.

We note that the gauge invariant term $|u|^2 u$ was not included in the previous works. Our purpose is to investigate the influence of the resonance term $|u|^2 u$ on the large time asymptotic behavior of solutions of nonresonance equation (2.1) with $\lambda_2 = \lambda_3 = 0$.

3. Main result

Our main result is the following. Denote $v_0(\xi) = ie^{-\frac{i}{2}\xi^2} u_0(\xi)$, $w_0(\xi) = v_0'(\xi) + \phi(\xi) v_0^3(\xi)$, and

$$\phi(\xi) = \frac{2\lambda}{\sqrt{3}} e^{-\frac{i}{2}\xi^2} \int_0^\xi \left(e^{\frac{i}{2}\eta^2} - \sqrt{3} e^{3\frac{i}{2}\eta^2} \right) d\eta.$$

Theorem 3.1. *Let $0 < |\lambda| < \sqrt{3}$. Let the initial data $v_0 \in \mathbf{H}^2$ and the norm $\|v_0\|_{\mathbf{L}^\infty} \leq \epsilon$ and $\|w_0\|_{\mathbf{H}^1} \leq \epsilon^4$, where $\epsilon > 0$ is sufficiently small. Also suppose that*

$$|v_0(0)| \geq \delta, \tag{3.1}$$

where $\delta = \epsilon^{1+\nu}$ with small $\nu > 0$. Then there exists a unique solution $u \in \mathbf{C}\left([1,\infty); \mathbf{L}^2\right)$ of the Cauchy problem (1.1). Moreover the time decay estimates

$$\begin{aligned} C_1 \delta t^{-\frac{1}{2}} \left(1 + \epsilon^4 \log t\right)^{-\frac{1}{4}} &\leq \sup_{|x| \leq \sqrt{t}} |u(t,x)| \\ &\leq C_2 \epsilon t^{-\frac{1}{2}} \left(1 + \delta^4 \log t\right)^{-\frac{1}{4}} \end{aligned}$$

and

$$\sup_{|x| > \sqrt{t}} |u(t,x)| \leq C\epsilon t^{-\frac{1}{2}}$$

are valid for large t.

We do not know the last estimate is sharp or not. The estimate

$$\sup_{|x|>t} |u(t,x)| \simeq C\epsilon t^{-\frac{1}{2}}$$

is true.

Remark 3.1. *For example, we can choose the initial data as follows $u_0(\xi) = -ie^{\frac{i}{2}\xi^2} v_0(\xi)$ and*

$$v_0(\xi) = \frac{\delta}{\sqrt{1 + 2\delta^2 \int_0^\xi \phi(\eta)\, d\eta + \epsilon^{12}\xi^2}}.$$

Then

$$w_0(\xi) = -\frac{\epsilon^{12}\xi}{\delta^2} v_0^3(\xi)$$

satisfies the estimate

$$\|w_0\|_{\mathbf{H}^1} \leq C\epsilon^6 \delta \left\| \left(1 + \epsilon^{12}\xi^2\right)^{-1} \right\|_{\mathbf{L}^2} \leq C\epsilon^3 \delta \leq \epsilon^4.$$

Remark 3.2. *Condition $|v_0(0)| \geq \delta$ of Theorem 3.1 excludes the vanishing condition at the origin for the final data $\widehat{u}_+(0) = 0$, which was assumed in papers [6], [7], [4], [5]. The condition $0 < |\lambda| < \sqrt{3}$ is essential, since we believe that the asymptotic behavior is different for the case of $|\lambda| \geq \sqrt{3}$.*

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