Four-Fermion Interaction Approximation of the Intermediate Vector Boson Model

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1 Introduction

In this note, we consider the approximation of the intermediate vector boson model at low energy by the four-fermion interaction model for the weak interaction of elementary particles. The intermediate vector boson model is described by the following Dirac-Proca equations in $3+1$ space time dimensions

$$i\gamma^\mu \partial_\mu \psi = \frac{1}{2}\gamma^\mu A_\mu (I-\gamma_5)\psi, \quad (t,x) \in \mathbf{R}^{1+3}, \tag{1}$$

$$\partial_\mu \partial^\mu A^\nu + M^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{1}{2}\langle \gamma^0 \gamma^\nu (I-\gamma_5)\psi, \psi\rangle, \quad (t,x) \in \mathbf{R}^{1+3}, \tag{2}$$

$$\psi(0,x) = \psi_0(x), \quad x \in \mathbf{R}^3, \tag{3}$$

$$A^\nu(0,x) = a^\nu(x), \quad \partial_0 A^\nu(0,x) = b^\nu(x), \quad x \in \mathbf{R}^3, \quad \nu = 0, \cdots, 3, \tag{4}$$

and the four-fermion interaction model is described by the following Dirac equation with cubic nonlinearity

$$i\gamma^\mu \partial_\mu \psi = \frac{1}{4M^2}\gamma^\mu \langle \gamma_0 \gamma_\mu (I-\gamma_5)\psi, \psi\rangle (I-\gamma_5)\psi, \quad (t,x) \in \mathbf{R}^{1+3}, \tag{5}$$

$$\psi(0,x) = \psi_0(x), \quad x \in \mathbf{R}^3, \tag{6}$$

where $M > 0$, I is the 4×4 identity matrix, γ^μ, $\mu = 0, \cdots, 3$ are 4×4 matrices satisfying the relations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$and $(\gamma^\mu)^* = g_{\mu\nu}\gamma^\nu$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

and $\langle z, w\rangle = \sum_{j=1}^{4} z_j \overline{w_j}$ for $z = (z_1, \cdots, z_4)$, $w = (w_1, \cdots, w_4) \in \mathbf{C}^4$. Here and hereafter, we follow the convention that Greek indices take values in $\{0, 1, 2, 3\}$ while Latin indices are valued in $\{1, 2, 3\}$. Indices repeated are summed. The space $\mathbf{R}^{1+3}$ is the four dimensional Euclidean space equipped with the flat Minkowski metric $(g_{\mu\nu}) = diag(1, -1, -1, -1)$. Indices are raised and lowered using the metric $g^{\mu\nu}$ and its inverse $g_{\mu\nu}$. We put $x^0 = t$ and $\partial_\mu = \partial/\partial x^\mu$.

ψ is a $\mathbf{C}^4$ valued function and A^μ, $\mu = 0, \cdots, 3$ are real-valued functions representing vectors in $\mathbf{R}^{1+3}$. The function ψ denotes the field of massless fermion with spin 1/2 and the functions A^μ, $\mu = 0, \cdots, 3$ denote the field of massive boson with spin 1. Equations (1) and (2) are called the Dirac and the Proca equations, respectively. This system appears in the intermediate vector boson model for the weak interaction of elementary particles before the adoption of the unified theory of electro-weak interactions (see [1, Section 10.1], [23] and [21]). Equation (5) is the Dirac equation with cubic nonlinearity, which corresponds to the four-fermion interaction. If the mass M of the intermediate vector boson is sufficiently large (or the energy of the boson is small relatively to the squared mass M^2), one may think that the system (1) and (2) is well approximated by equation (5) (see [1, Section 10.5] and [24, Section 8.3.2]). This is called the Fermi theory which gives an excellent description of four-fermion coupling phenomena involving the weak interaction at low energy such as the beta decay of a neutron into a proton, an electron and an electron antineutrino. In this note, we consider justifying this approximation of the intermediate vector boson model at low energy by the four-fermion interaction in a mathematically rigorsous sense.

If ψ and A^ν satisfy (1), then it is easily verified that equation (2) is equivalent to the following:

$$\Box A^\nu + M^2 A^\nu = \frac{1}{2}\langle \gamma^0\gamma^\nu(I - \gamma_5)\psi, \psi\rangle, \quad (t, x) \in \mathbf{R}^{1+3}, \tag{7}$$

$$\partial_\mu A^\mu = 0, \quad (t, x) \in \mathbf{R}^{1+3}, \tag{8}$$

where $\Box = \partial_\mu\partial^\mu$. Indeed, we take the derivatives in x^ν of (2) and sum the resulting equations over ν to obtain (8), under which equation (2) is equivalent to (7). The constraint (8) is called the Lorenz gauge condition, which used to be called the "Lorentz" gauge condition because of the confusion between Ludvig Lorenz and Hendrik Lorentz. Accordingly, the initial data (a^ν, b^ν) are chosen so that they satisfy (8) for $t = 0$. In that case, the constraint (8) is automatically satisfied as long as the solutions A^ν exist. Thus, the Cauchy problem (1)-(4) is reduced to (1), (3), (4) and (7) with (8) at $t = 0$.

Remark 1.1. We should more precisely state the relation between the Dirac-Proca equations (1), (7) and the Lorenz gauge condition (8). If (ψ, A^ν) are solutions of the Cauchy problem (1), (7), (8) and (3), (4), then the initial data (ψ_0, a^ν, b^ν) must satisfy the following compatibility condition relevant to the Lorenz gauge.

$$b^0 + \partial_j a^j = 0, \tag{9}$$

$$\Delta a^0 - M^2 a^0 + \frac{1}{2}\langle (I - \gamma_5)\psi_0, \psi_0\rangle + \partial_j b^j = 0. \tag{10}$$

Conversely, if the initial data (ψ_0, a^ν, b^ν) satisfy the above gauge constraint at $t = 0$, then the solutions A^ν, $0 \le \nu \le 3$ for (1), (7) and (4) automatically satify the Lorenz gauge condition (8) for all times. Therefore, after we have imposed the above gauge constraint on the initial data, we do not have to consider the Lorenz gauge condition (8).

We conclude this section with several notations given. For a Banacha space X and a nonnegative number k, let $o_X(M^{-k})$ and $O_X(M^{-k})$ denote various terms of smaller order and of not larger order, respectively, than M^{-k} as $M \to \infty$. Let $U_D(t)$ denote the evolution operator of the free Dirac equation.

2 Theorem and Sketch of Proof

In this section, we state our theorem concerning the justification of the four-fermion interaction approximation and give a sketch of the proof.

Theorem 2.1. *Let ε be any small positive constant. Assume that $(\psi_0, a_M^\nu, b_M^\nu) \in H^{1+\varepsilon} \times H^{1+\varepsilon} \times H^\varepsilon$, guage compatibility conditions* (9) *and* (10) *hold and the initial data (a_M^ν, b_M^ν) satisfy*

$$\|a_M^\nu\|_{H^{1+\varepsilon}} = o\big(M^{-1}\big), \qquad \|b_M^\nu\|_{H^\varepsilon} = o\big(1\big) \qquad (M \to \infty). \tag{11}$$

Then, there exist a positive constant T independent of M and unique solutions (ψ, A^ν) of (1), (3), (4), (7) *and* (8) *such that*

$$\psi \in C([-T,T]; H^{1+\varepsilon}), \quad A^\nu \in C([-T,T]; H^{1+\varepsilon}) \cap C^1([-T,T]; H^\varepsilon), \tag{12}$$

$$\|\psi\|_{C([-T,T];H^{1+\varepsilon})} = O\big(1\big), \quad \|A^\nu\|_{C([-T,T];H^{1+\varepsilon})} = o\big(1\big) \quad (M \to \infty) \tag{13}$$

and

$$\psi(t) = U_D(t)\psi_0 - \frac{i}{4M^2}\int_0^t U_D(t-s)\gamma^0\gamma^\mu\langle\gamma_0\gamma_\mu(I-\gamma_5)\psi,\psi\rangle(I-\gamma_5)\psi(s)\,ds \quad (14)$$
$$+o_{H^\varepsilon}\big(M^{-2}\big)\ \textit{uniformly on}\ [-T,T] \quad (M\to\infty).$$

Remark 2.2. (i) Formula (14) may be thought of as the four-fermion interaction approximation of the intermedate vector boson model at low energy. The coefficient before the integral on the right hand side of (14) is called the Fermi constant and it is known that it has dimensions of inverse squared mass. Formula (14) agrees with this physical observation. Theorem 2.1 implies that the solutions (ψ, A^μ) converge to $(U_D(t)\psi_0, 0)$ as $M \to \infty$. That is, four fermions get to have almost no interaction with each other as the mass of intermediate boson increases. This is why the weak interaction is so weak at low energy (see [1, the discussion from line 5 on page 342 to line 3 on page 343] and [24, Section 6.1.2 on pages 103 and 104]).

(ii) Assumption (11) seems to be natural, because in the physics context, it is always assumed that the kinetic energy part $(\partial_t^2-\Delta)A^\nu$ is much smaller than the rest energy part M^2A^ν at low energy. This and (7) suggest that if the solution ψ of the Dirac equation has the size independent of M, then a^ν should be $O(M^{-2})$ and $\partial_t^2 A^\nu(0)$ should be $O(1)$ $(M\to\infty)$, which aslo requires, together with interpolation, that b^ν should be $O(M^{-1})$ $(M\to\infty)$. For example, we note that for $u \in W^{2,2}(-T,T;L^2(\mathbf{R}^3))$, we have by integration by parts

$$\big\|\sqrt{\varphi}\partial_t u\big\|^2_{L^2([-T,T]\times\mathbf{R}^3)} \le C\big(\|u\|^2_{L^2([-T,T]\times\mathbf{R}^3)} + \|\partial_t^2 u\|_{L^2([-T,T]\times\mathbf{R}^3)}\|u\|_{L^2([-T,T]\times\mathbf{R}^3)}\big),$$

where $\varphi(t)$ is a time cut-off function in $C^\infty(\mathbf{R})$ such that $\varphi(t) = 0$ $(|t| \ge 4T/5)$, $\varphi(t) = 1$ $(|t| \le T/2)$ and $0 \le \varphi \le 1$.

(iii) In order to show the existence time T is independent of M, in Theorem 2.1 we have only to assume the following weaker condition than (11):

$$\|a_M^\nu\|_{H^{1+\varepsilon}} = O\big(M^{-1/2}\big), \qquad \|b_M^\nu\|_{H^\varepsilon} = O\big(1\big) \qquad (M\to\infty). \quad (15)$$

The local existence of solutions (ψ, A^ν) of (1), (3), (4), (7) and (8) follows immediately from the Stricharz estimate (see [9, Proposition 3.1 on pages 58-59], [20, Theorem4.2 on pages 99-100], [10, Corollary1.3 on pages 957-958] and [11, Lemma 5 on page 296]). We now see how the Strichartz estimate depends on M for the Klein-Gordon equation with mass M. Let N denote a dyadic number and let P_N be the Littlewood-Paley operator. We put $\langle\nabla\rangle_M = \mathcal{F}^{-1}(|\xi|^2+M^2)^{1/2}\mathcal{F}$ and $\langle N\rangle_M = (|N|^2+M^2)^{1/2}$. For the d-dimensional case, we have by the stationary phase method

$$\left\|e^{it\langle\nabla\rangle_M}P_N f\right\|_{L^\infty(\mathbf{R}^d)} \lesssim N\langle N\rangle_M^{(d-1)/2}|t|^{-(d-1)/2}\|P_N f\|_{L^1(\mathbf{R}^d)}$$

(see, e.g., [11, (2.4) on page 296], [8, (1.2) on page 344] and [14]), where the implicit constant is independent of M. We note that when $d=3$, the power $(d-1)/2$ of $\langle N\rangle_M$ on the right hand side is equal to 1. Therefore, we can conclude that when $d=3$, the factor $\langle N\rangle_M$ is harmless, because the operator that we have to estimate is not $e^{it\langle\nabla\rangle_M}$ but $e^{it\langle\nabla\rangle_M}/\langle\nabla\rangle_M$. We only use the Strichartz estimate which does not depend on M and so the existence time T can be chosen independent of M. The proof of the local existence is the same as that in [8] and [18].

We next breifly explain how to derive the formula (14). The Cauchy problem (1), (7) and (3), (4) can be rewritten as follows.

$$\psi(t) = U_D(t)\psi_0 - \frac{i}{2}\int_0^t U_D(t-s)\gamma^0\gamma^\mu A_\mu(s)(I-\gamma_5)\psi(s)\,ds, \tag{16}$$

$$\begin{aligned} A^\mu(t) =& \cos t\langle\nabla\rangle_M a_M^\mu + \frac{\sin t\langle\nabla\rangle_M}{\langle\nabla\rangle_M}b_M^\mu \\ &+\frac{1}{2}\int_0^t \frac{\sin(t-s)\langle\nabla\rangle_M}{\langle\nabla\rangle_M}\langle\gamma^0\gamma^\mu(I-\gamma_5)\psi(s),\psi(s)\rangle\,ds. \end{aligned} \tag{17}$$

By applying the integration by parts to the integral on the right hand side of (17), we have

$$\begin{aligned} A^\mu(t) =& \cos t\langle\nabla\rangle_M a_M^\mu + \frac{\sin t\langle\nabla\rangle_M}{\langle\nabla\rangle_M}b_M^\mu \\ &+\frac{1}{2}\left[\frac{\cos(t-s)\langle\nabla\rangle_M}{\langle\nabla\rangle_M^2}\langle\gamma^0\gamma^\mu(I-\gamma_5)\psi(s),\psi(s)\rangle\right]_{s=0}^{s=t} \\ &-\frac{1}{2}\int_0^t \frac{\cos(t-s)\langle\nabla\rangle_M}{\langle\nabla\rangle_M^2}\partial_0\langle\gamma^0\gamma^\mu(I-\gamma_5)\psi(s),\psi(s)\rangle\,ds \\ =& \cos t\langle\nabla\rangle_M a_M^\mu + \frac{\sin t\langle\nabla\rangle_M}{\langle\nabla\rangle_M}b_M^\mu + \frac{1}{2M^2}\langle\gamma^0\gamma^\mu(I-\gamma_5)\psi,\psi\rangle \\ &+\frac{1}{2}\Big(\frac{1}{\langle\nabla\rangle_M^2}-\frac{1}{M^2}\Big)\langle\gamma^0\gamma^\mu(I-\gamma_5)\psi,\psi\rangle - \frac{\cos t\langle\nabla\rangle_M}{2\langle\nabla\rangle_M^2}\langle\gamma^0\gamma^\mu(I-\gamma_5)\psi_0,\psi_0\rangle \\ &-\frac{1}{2}\int_0^t \frac{\cos(t-s)\langle\nabla\rangle_M}{\langle\nabla\rangle_M^2}\partial_0\langle\gamma^0\gamma^\mu(I-\gamma_5)\psi(s),\psi(s)\rangle\,ds. \end{aligned} \tag{18}$$

When we insert the right hand side of (18) into (16), the third term on the right hand side of (18) corresponds to the second term on the right hand side of (14) and the rest terms on the right hand side of (18) can be regarded as smaller order terms than M^{-2} as $M \to \infty$. This is because the fourth term on the right hand side of (18) includes the factor $\left(\frac{1}{\langle\nabla\rangle_M^2} - \frac{1}{M^2}\right)$ and the fifth and the sixth terms include the rapid oscillation factors as $M \to \infty$, which leads to extra decay as $M \to \infty$. Furthermore, the assumption (11) implies that the time integral of the first and the second terms on the right hand side of (18) is $o_{H^\varepsilon}(M^{-2})$ as $M \to \infty$.

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