On viscosity solutions of parabolic equations on the Heisenberg group

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1 Introduction

There has been a recent trend of extending the Euclidean viscosity solution theory for fully nonlinear PDEs from the Euclidean space to general metric spaces due to many important applications such as optimal transport etc. While the first order case (eg. Hamilton-Jacobi equations) is comparatively easier to handle, second order problems turn out to be more challenging. As one of the explicit examples of metric spaces that have very different structures from the Euclidean spaces, the Heisenberg group (or a more general sub-Riemannian manifold) is often regarded as a testing ground when developing a second order theory.

In this talk, we discuss some nonlinear parabolic equations on the Heisenberg group $\mathcal{H}$. Our main interest lies in the level set formulation of the horizontal mean curvature flow, for which we prove uniqueness, existence and stability of its axisymmetric viscosity solutions. Motivated by the well known behavior of usual mean curvature flow, we also study the preserving properties of Lipschitz continuity and convexity for parabolic equations on the Heisenberg group but in a simpler semi-linear case.

The talk is based on a joint work with F. Ferrari and J. J. Manfredi [2] and a more recent work with J. J. Manfredi and X. Zhou [5].

2 Preliminaries on the Heisenberg group

For convenience of audience, we present a brief introduction on the Heisenberg group in what follows. More details are referred to [1] and [6]. Recall that the Heisenberg group $\mathcal{H}$ is $\mathbb{R}^3$ endowed with the non-commutative group multiplication

$$(p_1, p_2, p_3) \cdot (q_1, q_2, q_3) = \left(p_1 + q_1, p_2 + q_2, p_3 + q_3 + \frac{1}{2}(p_1 q_2 - q_1 p_2)\right),$$

for all $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2 . q_3)$ in $\mathcal{H}$. Note that the group inverse of $p = (p_1, p_2, p_3)$ is $p^{-1} = (-p_1, -p_2, -p_3)$. The Korányi gauge is given by

$$|p| = ((p_1^2 + p_2^2)^2 + 16 p_3^2)^{1/4},$$

and the left-invariant Korányi or gauge metric is $d(p, q) = |q^{-1} \cdot p|$.

The Korányi ball of radius $r > 0$ centered at p is $B_r(p) := \{q \in \mathcal{H}\colon d(p,q) < r\}$. The Lie algebra of $\mathcal{H}$ is generated by the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial p_1} - \frac{p_2}{2}\frac{\partial}{\partial p_3}, \quad X_2 = \frac{\partial}{\partial p_2} + \frac{p_1}{2}\frac{\partial}{\partial p_3}, \quad X_3 = \frac{\partial}{\partial p_3}.$$

One may easily verify the commuting relation $X_3 = [X_1, X_2] = X_1X_2 - X_2X_1$.

For any smooth real valued function u defined in an open subset of $\mathcal{H}$, the horizontal gradient of u is $\nabla_H u = (X_1 u, X_2 u)$. The symmetrized horizontal Hessian $(\nabla_H^2 u)^*$ is the 2×2 symmetric matrix given by

$$(\nabla_H^2 u)^* := \begin{pmatrix} X_1^2 u & (X_1X_2u + X_2X_1u)/2 \\ (X_1X_2u + X_2X_1u)/2 & X_2^2 u \end{pmatrix}.$$

3 Horizontal Mean Curvature Flow

We are interested in a family of compact hypersurfaces $\{\Gamma_t\}_{t \geq 0}$ in $\mathcal{H}$ parametrized by time $t \geq 0$. The motion of the hypersurfaces is governed by the following law:

$$V_H = \kappa_H, \tag{1}$$

where V_H denotes its *horizontal normal velocity* and κ_H stands for the *horizontal mean curvature*. By using a level set formulation, we study the following PDE problem:

$$\text{(MCF)} \quad \begin{cases} u_t - \operatorname{tr}\left[\left(I - \dfrac{\nabla_H u \otimes \nabla_H u}{|\nabla_H u|^2}\right)(\nabla_H^2 u)^*\right] = 0 & \text{in } \mathcal{H} \times (0, \infty), \quad (2) \\ u(p, 0) = u_0(p) & \text{in } \mathcal{H}. \quad (3) \end{cases}$$

with a given function $u_0 \in C(\mathcal{H})$ satisfying

$$\Gamma_0 = \{p \in \mathcal{H} : u_0(p) = 0\}.$$

Our main result is as follows.

Theorem 1. *Assume that u_0 is a Lipschitz continuous function in $\mathcal{H}$ and is constant $C \in \mathbb{R}$ outside a compact set. Then there exists a unique axisymmetric viscosity solution of (MCF) in $\mathcal{H} \times [0, \infty)$.*

The proof of uniqueness is based on a comparison principle adapted to sub-Riemannian structures. We need the symmetry assumption to overcome the singularity in (2) when $\nabla_H u = 0$. The existence of solutions is obtained via a game theoretic approximation, following the idea in [4] in the Euclidean space.

Our uniqueness and existence results enable us to discuss motion by mean curvature with a variety of initial hypersurfaces including spheres, tori and other compact surfaces. We are particularly interested in the motion of a subelliptic sphere. It turns out that if u_0 is a defining function of the sphere centered at 0 with radius r, say

$$u_0(p) = \min\{(p_1^2 + p_2^2)^2 + 16p_3^2 - r^4, M\}$$

with $p = (p_1, p_2, p_3) \in \mathcal{H}$ and $M > 0$ large, then the unique solution of (MCF) is

$$u(p,t) = \min\{(p_1^2 + p_2^2)^2 + 12t(p_1^2 + p_2^2) + 16p_3^2 + 12t^2 - r^4, M\}$$

for any $t \geq 0$. It is obvious that the zero level set Γ_t of u vanishes after time $t = r^2/\sqrt{12}$, which, by comparison principle, indicates that all compact surfaces under the motion by horizontal mean curvature disappear in finite time.

4 Lipschitz and convexity preserving

We are also interested in the Lipschitz and convexity preserving properties for (MCF). In $\mathbb{R}^n$, Lipschitz continuity and convexity preserving are important properties, which hold for a large class of nonlinear parabolic equations: when the initial value u_0 is Lipschitz continuous (resp., convex), the unique solution $u(x,t)$ is Lipschitz (resp., convex) in x as well for any $t \geq 0$; we refer the reader to [3] for more details in the Euclidean space.

Although the notions of (left-invariant) Lipschitz continuity and convexity in $\mathcal{H}$ are available in the literature, extension of these preserving properties to the Heisenberg group is not immediate. Methods in the Euclidean case are not applicable directly in $\mathcal{H}$, since the mixed second derivatives in $\mathcal{H}$ are not commutative in general. In fact, our counterexamples show that these properties do not hold in general even for the following semi-linear equation:

$$\begin{cases} u_t - \operatorname{tr}(A(\nabla_H^2 u)^*) + f(\nabla_H u) = 0 & \text{in } \mathcal{H} \times (0, \infty), \quad (4) \\ u(\cdot, 0) = u_0 & \text{in } \mathcal{H}, , \quad (5) \end{cases}$$

where A is a given 2×2 symmetric positive-semidefinite matrix and the function $f : \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies certain regularity assumptions.

We show that the unique solution of (4)–(5) only preserves right-invariant Lipschitz continuity and convexity. We then give several additional assumptions, under which the preserving of left-invariant Lipschitz continuity and convexity can be proved.

References

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