p-ラプラシアンを含む精密化された加藤の不等式とその応用

劉 暁静 (Xiaojing Liu), 堀内　利郎 (Toshio HORIUCHI)
Ibaraki University, Mito, Ibaraki, Japan.

平成 28 年 12 月 16 日

1 Introduction

Ω : a bounded domain of $\mathbf{R}^N$ $(N \geq 1)$.

1.1 The Classical Convex type inequality

Lemma 1 *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ s.t. $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, then $\Delta|u|$ and Δu^+ are Radon measures and we have*

$$\Delta|u| \geq \mathrm{sgn}(u)\Delta u \qquad \text{in } D'(\Omega), \tag{1}$$

$$\Delta u^+ \geq \chi_{[u\geq 0]}\Delta u \qquad \text{in } D'(\Omega), \tag{2}$$

where $\mathrm{sgn}(s) = 1$ *if* $s > 0$, -1 *if* $s < 0$ *and zero at* $s = 0$ $u^+ = \max[u, 0]$.

1.2 The classical Concave type inequality

Definition 1 (Truncation) *: $T_k(s)$: Given $k > 0$, we denote by T_k:$\mathbf{R} \to \mathbf{R}$ a truncation function*

$$T_k(s) := k,\ \text{if } s \geq k; s,\ \text{if } -k < s < k; -k,\ \text{if } s \leq -k. \tag{3}$$

Lemma 2 *Assume that $u \in L^1_{\mathrm{loc}}(\Omega)$, $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$ and $u \geq 0$ a.e. in Ω. Then, for any $k \geq 0$ we have*

$$\Delta(T_k(u)) \leq \chi_{[0\leq u\leq k]}\Delta u \quad \text{in } D'(\Omega), \tag{4}$$

where $\chi_S(x)$ is a characteristic function of $S \subset \mathbf{R}$.

Moreover, whenΔu can be replaced by $\Delta_p u$ under additional assumptions on distributional derivatives of $u \in L^1_{\mathrm{loc}}(\Omega)$. Here, p-Laplace operator is defined by $\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$.

2 Main Aim

Consider a class of second order elliptic operators $\mathcal{A}$ including Δ_p and establish improved Kato's inequalities when $\mathcal{A}u$ is a Radon measure.

$$\mathcal{A}u = \mathrm{div}\, A(x, \nabla u), \tag{5}$$

where $A : \Omega \times R^N \mapsto R^N$ satisfies the following assumptions for some positive numbers c_1, c_2 and c_3:

1. The function $x \mapsto A(x, \xi)$ is bounded measurable for $^\forall\xi \in R^N$,
2. The function $\xi \mapsto A(x, \xi)$ is continuous for a.e. $x \in \Omega$,
3. $$|A(x,\xi) - A(x,\eta)| \leq c_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|, \quad ^\forall\xi, \eta \in R^N,\ \text{a.e. } x \in \Omega,$$
4. $$(A(x,\xi) - A(x,\eta)) \cdot (\xi - \eta) \geq c_3(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2,\ ^\forall\xi, \eta \in R^N,\ \text{a.e. } x \in \Omega,$$
5. $$A(x, \lambda\xi) = \lambda|\lambda|^{p-2}A(x, \xi), \quad \text{for all } \lambda \in R, \lambda \neq 0.$$

Definition 2 *($M(\Omega)$: the space of Radon measure):*
$\mu \in M(\Omega) \iff$ *For every open set $\omega \subset\subset \Omega$, $^\exists C_\omega > 0$ s.t. $|\int_\Omega \varphi\, d\mu| \leq C_\omega ||\varphi||_{L^\infty}$, for $^\forall\varphi \in C_0^\infty(\omega)$.*

3 Decomposition of Radon measures

For any $\mu \in M(\Omega)$, μ can be uniquely decomposed as a sum of two Radon measures on Ω (see e.g. [6, 9]) : $\mu = \mu_d + \mu_c$, where

$$\begin{cases} \mu_d(A) = 0 \quad \text{for} \quad \text{any Borel set } A \subset \Omega \text{ s.t } C_p(A,\Omega) = 0, \\ |\mu_c|(\Omega \setminus F) = 0 \quad \text{for} \quad \text{some Borel set } F \subset \Omega \text{ s.t } C_p(F,\Omega) = 0. \end{cases}$$

Definition 3 *(A p-capacity relative to Ω) For each compact set $K \subset \Omega$,*
$C_p(K,\Omega) = \inf\{\int_\Omega |\nabla\varphi|^p : \varphi \in C_0^\infty(\Omega), \varphi \geq 1$ *in some nbd of* $K\}$.

4 Definition of admissible class

Definition 4 **(Admissible class in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$)** *Let $p^* = \max(1, p-1)$. A function $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ is said to be admissible iff $\mathcal{A}u \in M(\Omega)$ and there exists a sequence $\{u_n\}_{n=1}^\infty \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ s.t:*

1. *$u_n \to u$ a.e. in Ω, $u_n \to u$ in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ as $n \to \infty$.*
2. *$\mathcal{A}u_n \in L^1_{\mathrm{loc}}(\Omega)$ $(n = 1, 2, \cdots)$ and*

$$\sup_n |\mathcal{A}u_n|(\omega) = \sup_n \int_\omega |\mathcal{A}u_n| < \infty \qquad \text{for every } \omega \subset\subset \Omega. \tag{6}$$

5 Some results on the admissibility

1. If $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ is admissible $\Longrightarrow$ $u^+ = \max[u, 0]$, $\quad u^- = \max[-u, 0]$, $\quad T_k(u)$ are admissible.
2. $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for $^\forall k > 0$. Moreover, given $\omega \subset\subset \omega' \subset\subset \Omega$, $^\exists C > 0$ independent on u s.t

$$\int_\omega |\nabla T_k(u)|^p \leq Ck \left(\int_{\omega'} |\Delta_p u| + \int_{\omega'} |\nabla u|^{p-1} \right)$$

3. When $p = 2$ and $\mathcal{A} = \Delta$, $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $\Delta u \in M(\Omega)$ $\Longrightarrow$ u is admissible.
4. $u \in W^{1,p}_0(\Omega)$, $\mathcal{A}u \in M(\Omega)$ $\Longrightarrow$ u is admissible.

6 Counter-example due to J.Serrin

Let Ω be a unit ball $B_1 = \{x \in R^N : |x| < 1\}$, and set

$$a_{i,j} = \delta_{i,j} + (a-1)\frac{x_i x_j}{r^2}, \qquad (r = |x|), \qquad \mathcal{B}u = \sum_{j,k=1}^N \frac{\partial}{\partial x_j}\left(a_{j,k}(x)\frac{\partial u}{\partial x_k} \right) = 0. \tag{7}$$

$$U(x) = x_1 r^{-\alpha}, \qquad \text{where} \quad \alpha = \frac{N}{2} + \sqrt{\left(\frac{N}{2} - 1\right)^2 + \frac{N-1}{a}}. \tag{8}$$

Proposition 1 *Assume that $a > 1$. Then $U \in W^{1,1}_{\mathrm{loc}}(B_1)$ and $\mathcal{B}U = 0$ in $D'(B_1)$. But U is not admissible, and $\mathcal{B}(U^+)$ is not a Radon measure. (Note that If $a > 1 \Longrightarrow N - 1 < \alpha < N$.)*

7 Main results and Applications

7.1 Improved Concave type inequality

Theorem 1
Assume $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ and u is addmissible. If $u \geq 0$ a.e. in Ω, then $\Delta_p(T_k(u))$ is a Radon measure for every $k > 0$, and

$$\Delta_p(T_k(u)) \leq (\Delta_p u)^+. \tag{9}$$

7.2 Application to Strong Maximum Principle

Theorem 2 *Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$, $u \geq 0$ a.e. and u is admissible. Then*

1. *There exists a quasicontinuous function (w.r.t. C_p) $\tilde{u} : \Omega \mapsto \mathbf{R}$ such that $u = \tilde{u}$ a.e. in Ω.*
2. *Assume that $-\Delta_p u \geq 0$ in Ω in the sense of measures. If $\tilde{u} = 0$ on some $K \subset \Omega$ with $C_p(K, \Omega) > 0$, then $u = 0$ a.e. in Ω.*

Remark 7.1 *$-\Delta_p u$ can be replaced by $-\Delta_p u + au^q$, where $0 \leq a \in L^1_{loc}(\Omega)$ and $q \geq p - 1$.*

7.3 Convex type Kato's inequality

Theorem 3 *Let Φ be a C^1 convex function s.t $0 \leq \Phi' < \infty$. Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ and u is addmissible. Then*

$$\Delta_p \Phi(u) \geq \Phi'(u)^{p-1}(\Delta_p u)_d - ||\Phi'||_{L^\infty(\mathbf{R})}(\Delta_p u)^-_c \quad \text{in} \quad D'(\Omega). \tag{10}$$

Corollary 1

$$\Delta_p(u^+) \geq \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)^-_c \qquad \text{in } D'(\Omega), \tag{11}$$

$$\Delta_p|u| \geq \operatorname{sgn}(u)(\Delta_p u)_d - |\Delta_p u|_c \qquad \text{in } D'(\Omega). \tag{12}$$

Example 1 *$u = |x|^\alpha$ for $\alpha = (p - N)/(p - 1)$ and $0 \in \Omega$.*

1. *u satisfies $\Delta_p u = \alpha|\alpha|^{p-2}c_N\delta$, If $p > 2 - 1/N$, then $|\nabla u| \in L^1_{loc}(\Omega)$ and u is addimible.*
2. *$\Delta_p(u^+) = \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)^-_c \qquad inD'(\Omega)$.*

7.4 Inverse maximum principle

Theorem 4 (**IMP**) *Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$,$u \geq 0$ and u is admissible. Then we have*

$$(-\Delta_p u)_c \geq 0 \quad \text{on } \Omega. \tag{13}$$

Corollary 2 *Assume the same assumptions in Theorem 3. Then, $(-\Delta_p(u^+))_c = (-\Delta_p u)^+_c$ on Ω.*

Theorem 5 *(Application of IMP) Suppose that u is admissible. Then $\operatorname{supp} \mu^\pm_c \subset \{x : u = \pm\infty\}$ for $\mu = -\Delta_p u$.*

Remark 7.2 *If $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ is an admissible solution of of $-\Delta_p u = \mu \in M(\Omega)$, then u is also a (local) renormalized solution of $-\Delta_p u = \mu$.*

8 Existence of admissible solution

Theorem 6 *Assumethat $\mu \in M(\Omega)$ and $|\mu|(\Omega) < \infty$. Then*

$$\begin{cases} -\Delta_p u = \mu, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \Omega. \end{cases} \tag{14}$$

has an admissible solution in $W^{1,p^*}_0(\Omega)$.

Lemma 3 *Let $\{\mu_n\}$ satisfy $\sup_n |\mu_n|(\Omega) < \infty$ and $\{u_n\}$ be admissible. Assume that*

$$\begin{cases} -\Delta_p u_n = \mu_n, & \text{in } \Omega, \\ \quad u_n = 0, & \text{on } \Omega. \end{cases} \tag{15}$$

holds for $n \in \mathbf{N}$. Then, up to a subsequence, $u_n \to {}^\exists u \in W^{1,p^*}_0(\Omega)$ s.t. u is admissible and satisfy (14) for ${}^\exists\mu$.

参考文献

[1] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.L. Vazquez, An L^1- theory of existence and uniqueness of solutions of nonlinear elliptic equations, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 22, No.2, (1995), 241-273.

[2] P. Bénilan, H. Brezis, Nonlinear problems related to the Thomas-Fermi equation, J. Evol. Equ. 3, (2004), 673-770.

[3] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure deta, J. Funct. Anal. vol. 87, 1989, 149-169

[4] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Annales de l'I.H.P. section C, tome 13 no. 5 (1996), p. 539-551.

[5] H. Brezis , A. Ponce, Remarks on the strong maximum principle, Differential Integral Equations 16 (2003), 1-12.

[6] H. Brezis, A. Ponce, Kato's inequality when Δu is a measure, C. R. Acad. Sci. Paris, Ser. I 338 (2004),599-604.

[7] H. Brezis, M. Marcus, A. Ponce, Nonlinear elliptic equations with measures revisited, in Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, and S. Klainerman, eds.), Annals of Mathematics Studies, 163 Princeton University Press, Princeton, NJ 2007, p. 55-110.

[8] Lorenzo D'Ambrosio, Enzo Mitidieri, A priori estimates and reduction principles for quasilinear elliptic problems and applications, Adv. Differential Equations 17 (2012), no. 9- 10, 935 20131000.

[9] L. Dupaigne and A. Ponce, Singularities of positive supersolutions in elliptic PDEs, Selecta Math. (N.S.) 10, (2004), 341-358.

[10] T. Horiuchi, Some remarks on Kato's inequality, J. of Inequal. & Appl., vol. 6, (2001), 29-36.

[11] T. Horiuchi, Kato's Inequalities for Degenerate Quasilinear Elliptic Operators, Kyungpook Mathematical Journal 2008 Vol. 48, No. 1, 15—24

[12] T. Kato, Schrödinger operators with singular potentials.Israel J. Math.13 (1972), 135-148.

[13] X. Liu, T. Horiuchi, Remarks on the strong maximum principle involving p-Laplacian, Hiroshima Mathematical Journal Volume 46, No.3 (November 2016) or Volume 47, No.1 (March 2017)

[14] X. Liu, Toshio Horiuchi, Remarks on Kato's inequality when $\Delta_p u$ is a measure, to appear in Mathematical Journal of Ibaraki University Volume 48, P. 45-61, 2017

[15] F. Maeda, Renormalized solutions of Dirichlet problems for quasilinear elliptic equations with general measure data, Hiroshima Math. J., 38 (2008), 51-93.

[16] G.D. Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa, 28, (1999), 741-808457-468.

[17] M.F. Bidaut-Víron, M. Garcia-Huidobro, L. Víron, Remarks on some quasilinear equations with gradient terms and measure data, arXiv:1211.6542 [math.AP] 13 Feb (2013).

[18] Bidaut-Véron M.F., Nguyen Quoc, H. and L. Véron Quasilinear Emden-Fowler equations with absorption terms and measure data, preprint