CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS IN HIGHER SPACE DIMENSIONS

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We consider the initial value problem for the nonlinear Schrödinger equation

$$\left\{ \begin{array}{l} i\partial_t u + \frac{1}{2}\Delta u = \lambda \left| u \right|^{\frac{2}{n}} u, \ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n \end{array} \right. \tag{0.1}$$

in space dimensions $n \geq 4$, where $\lambda \in \mathbb{R}$. In the case of $1 \leq n \leq 3$, asymptotic behavior of small amplitude solutions to (0.1) has been studied in [1], [2], [3], [4], [5]. However in the case of $n \geq 4$, there are no results for asymptotic behavior of solutions as far as we know. Our purpose in this talk is to show the sharp asymptotics and time decay of solutions to (0.1) in the uniform norm for higher space dimensions $n \geq 4$.

We introduce some function spaces and notations. Let $\mathbf{L}^\infty \cap \mathbf{C}$ denote the bounded continuous function space with the norm $\|\phi\|_{\mathbf{L}^\infty \cap \mathbf{C}} = \sup_{x \in \mathbb{R}^n} |\phi(x)|$. The homogeneous Sobolev space $\dot{\mathbf{H}}^m$ is defined by

$$\dot{\mathbf{H}}^m = \left\{ \phi; \|\phi\|_{\dot{\mathbf{H}}^m} = \left\| (-\Delta)^{m/2} \phi \right\|_{\mathbf{L}^2} < \infty \right\},$$

$m \geq 0$, where $\|\phi\|_{\mathbf{L}^2}^2 = \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx$. Denote $\langle t \rangle = \sqrt{1+t^2}$. To state our results, we use the function space

$$\mathbf{X} = \left\{ u; \mathcal{F}\mathcal{U}(-t-1) u \in \mathbf{C}([0,\infty); \mathbf{Y}), \|u\|_{\mathbf{X}} < \infty \right\},$$

where $\mathbf{Y} = \mathbf{L}^\infty \cap \mathbf{C} \cap \dot{\mathbf{H}}^\sigma$, $\frac{n}{2} < \sigma < \frac{n}{2} + 1$ and

$$\begin{aligned} \|u\|_{\mathbf{X}} &= \sup_{0 \leq t < \infty} \|\mathcal{F}\mathcal{U}(-t-1) u(t)\|_{\mathbf{L}^\infty \cap \mathbf{C}} \\ &\quad + (t+1)^{-\gamma} \|\mathcal{F}\mathcal{U}(-t-1) u(t)\|_{\dot{\mathbf{H}}^\sigma} \end{aligned}$$

with a small γ satisfying $\frac{1}{n}\left(\sigma - \frac{n}{2}\right) > \gamma > 0$. We note here that the Hölder class of order $\sigma - \frac{n}{2}$ is included in $\mathbf{Y}$.

Theorem 0.1. *We assume that the initial data satisfy*

$$\frac{\rho}{2} \leq \inf_{\xi \in \mathbb{R}^n} |\widehat{u_0}(\xi)| \leq \|\widehat{u_0}\|_{\mathbf{L}^\infty \cap \mathbf{C}} \leq \rho$$

and $e^{\frac{i}{2}|\xi|^2}\widehat{u_0} \in \mathbf{L}^\infty \cap \mathbf{C} \cap \dot{\mathbf{H}}^\sigma$, $\|e^{\frac{i}{2}|\xi|^2}\widehat{u_0}\|_{\dot{\mathbf{H}}^\sigma} \leq \rho^2$ *with* $\frac{n}{2} < \sigma < \frac{n}{2} + 1$. *Then there exists a* $\rho_0 > 0$ *such that the Cauchy problem (0.1) has a unique solution* $u \in \mathbf{X}$ *for all* $0 < \rho \leq \rho_0$. *Moreover the time decay estimate*

$$\frac{1}{5}\rho(t+1)^{-\frac{n}{2}} \leq \inf_{x \in \mathbb{R}^n} |u(t)| \leq \|u(t)\|_{\mathbf{L}^\infty \cap \mathbf{C}} \leq \frac{7}{5}\rho(t+1)^{-\frac{n}{2}}$$

holds for all $t > 0$.

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Remark 0.1. *Typical example of the data could be the following*

$$\widehat{u_0}(\xi) = e^{-\frac{i}{2}|\xi|^2}\rho\left(1 - \frac{\rho^2}{\langle\xi\rangle}\right)$$

since by a direct calculation

$$\left\| e^{\frac{i}{2}|\xi|^2}\widehat{u_0}\right\|_{\dot{\mathbf{H}}^\sigma} = \left\|\rho\left(1 - \frac{\rho^2}{\langle\xi\rangle}\right)\right\|_{\dot{\mathbf{H}}^\sigma} \leq C\rho^3 \leq \rho^2$$

and

$$\rho - \rho^3 \leq \inf_{\xi\in\mathbb{R}^n}\left|\widehat{u_0}(\xi)\right| \leq \left\|\widehat{u_0}\right\|_{\mathbf{L}^\infty\cap\mathbf{C}} \leq \rho.$$

Theorem 0.2. *Let u be the solution constructed in Theorem 0.1. Then there exists a unique final state $\widehat{u_+} \in \mathbf{L}^\infty\cap\mathbf{C}\cap\dot{\mathbf{H}}^\beta$, $\frac{n}{2} < \beta < \sigma < \frac{n}{2}+1$, such that the asymptotics*

$$\left\| u(t) - e^{\frac{i|x|^2}{2(t+1)} - i\frac{n\pi}{4}}(t+1)^{-\frac{n}{2}} e^{-i\lambda\left|\widehat{u_+}\left(\frac{x}{t+1}\right)\right|^{\frac{2}{n}}\log(t+1)}\widehat{u_+}\left(\frac{x}{t+1}\right)\right\|_{\mathbf{L}^\infty\cap\mathbf{C}}$$
$$\leq C(t+1)^{-\frac{n}{2}-\frac{2}{n}(\delta-\gamma)}\left(\rho^2 + \rho^{\left(\frac{2}{n}+2\right)\frac{2}{n}+1}\log(t+1)\right)$$

holds for all $t > 0$ and

$$\frac{1}{5}\rho \leq \inf_{\xi\in\mathbb{R}^n}\left|\widehat{u_+}(\xi)\right| \leq \left\|\widehat{u_+}\right\|_{\mathbf{L}^\infty} \leq \frac{7}{5}\rho,$$

where $\delta \in \left(0, \frac{1}{2}\left(\sigma - \frac{n}{2}\right)\right)$, $0 < \gamma < \frac{\sigma-\beta}{n}$.

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