CRITICAL NONLINEAR SCHröDINGER EQUATIONS IN HIGHER SPACE DIMENSIONS

NAKAO HAYASHI

We consider the initial value problem for the nonlinear Schrödinger equation

\begin{equation}
\begin{aligned}
&i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^\frac{2}{n} u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\
u(0, x) = u_0(x), \quad x \in \mathbb{R}^n
\end{aligned}
\end{equation}

in space dimensions \( n \geq 4 \), where \( \lambda \in \mathbb{R} \). In the case of \( 1 \leq n \leq 3 \), asymptotic behavior of small amplitude solutions to (0.1) has been studied in [1], [2], [3], [4], [5]. However in the case of \( n \geq 4 \), there are no results for asymptotic behavior of solutions as far as we know. Our purpose in this talk is to show the sharp asymptotics and time decay of solutions to (0.1) in the uniform norm for higher space dimensions \( n \geq 4 \).

We introduce some function spaces and notations. Let \( L^1 \cap C \) denote the bounded continuous function space with the norm

\[
\|\phi\|_{L^1 \cap C} = \sup_{x \in \mathbb{R}^n} |\phi(x)|.
\]

The homogeneous Sobolev space \( \dot{H}^m \) is defined by

\[
\dot{H}^m = \left\{ \phi : \|\phi\|_{\dot{H}^m} = \left\| (-\Delta)^{m/2} \phi \right\|_{L^2} < \infty \right\},
\]

\( m \geq 0 \), where \( \|\phi\|_{L^2}^2 = \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx \). Denote \( (t) = \sqrt{1 + t^2} \). To state our results, we use the function space

\[
X = \{ u : \mathcal{F}U(-t-1) u \in C([0, \infty) ; Y), \|u\|_X < \infty \},
\]

where \( Y = L^\infty \cap C \cap \dot{H}^\sigma, \frac{\sigma}{2} < \sigma < \frac{n}{2} + 1 \) and

\[
\|u\|_X = \sup_{0 \leq t < \infty} \|\mathcal{F}U(-t-1)u(t)\|_{L^\infty \cap C}
\]

\[
+ (t+1)^{-\gamma} \|\mathcal{F}U(-t-1)u(t)\|_{\dot{H}^\sigma}
\]

with a small \( \gamma \) satisfying \( \frac{1}{2} \left( \sigma - \frac{n}{2} \right) > \gamma > 0 \). We note here that the Hölder class of order \( \sigma - \frac{n}{2} \) is included in \( Y \).

**Theorem 0.1.** We assume that the initial data satisfy

\[
\frac{\rho}{2} \leq \inf_{\xi \in \mathbb{R}^n} |\hat{u}_0(\xi)| \leq \|\hat{u}_0\|_{L^\infty \cap C} \leq \rho
\]

and \( e^{\xi |\xi|^2} \hat{u}_0 \in L^\infty \cap C \cap \dot{H}^\sigma, \|e^{\xi |\xi|^2} \hat{u}_0\|_{\dot{H}^\sigma} \leq \rho^2 \) with \( \frac{\sigma}{2} < \sigma < \frac{n}{2} + 1 \). Then there exists a \( \rho_0 > 0 \) such that the Cauchy problem (0.1) has a unique solution \( u \in X \) for all \( 0 < \rho \leq \rho_0 \). Moreover the time decay estimate

\[
\frac{1}{3} \rho(t+1)^{-\frac{n}{2}} \leq \inf_{x \in \mathbb{R}^n} |u(t)| \leq \|u(t)\|_{L^\infty \cap C} \leq \frac{7}{3} \rho(t+1)^{-\frac{n}{2}}
\]

holds for all \( t > 0 \).

**Key words and phrases.** This is a joint work with Chunhua Li and Pavel Naumkin.
Remark 0.1. Typical example of the data could be the following
\[ \bar{u}_0(\xi) = e^{-\frac{i}{2} \xi^2} \rho \left( 1 - \frac{\rho^2}{\xi} \right) \]
since by a direct calculation
\[ \left\| e^{\frac{i}{2} \xi^2} \bar{u}_0 \right\|_{\dot{H}^s} = \left\| \rho \left( 1 - \frac{\rho^2}{\xi} \right) \right\|_{\dot{H}^s} \leq C \rho^3 \leq \rho^2 \]
and
\[ \rho - \rho^3 \leq \inf_{\xi \in \mathbb{R}^n} |\bar{u}_0(\xi)| \leq ||\bar{u}_0||_{L^{\infty}\cap C} \leq \rho. \]

Theorem 0.2. Let \( u \) be the solution constructed in Theorem 0.1. Then there exists a unique final state \( \bar{u}_+ \in L^{\infty}\cap C \cap H^\beta, \frac{1}{2} < \beta < \frac{3}{2} + 1 \), such that the asymptotics
\[ \left\| u(t) - e^{\frac{i}{2} \xi^2} - i \frac{\alpha}{2} \left( t + 1 \right)^{-\frac{\alpha}{2}} e^{-i \alpha |\bar{u}_+| \left( t + 1 \right)^{\frac{\alpha}{2}}} \bar{u}_+ \left( \frac{x}{t + 1} \right) \right\|_{L^{\infty}\cap C} \]
\[ \leq C (t + 1)^{-\frac{\beta}{2} - \frac{2}{3}(\delta - \gamma)} \left( \rho^2 + \rho \left( \frac{\beta}{2} + 1 \right) + 1 \log (t + 1) \right) \]
holds for all \( t > 0 \) and
\[ \frac{1}{2} \rho \leq \inf_{\xi \in \mathbb{R}^n} |\bar{u}_+(\xi)| \leq ||\bar{u}_+||_{L^{\infty}} \leq \frac{7}{5} \rho, \]
where \( \delta \in \left( 0, \frac{1}{2} \left( \sigma - \frac{n}{2} \right) \right), 0 < \gamma < \frac{\sigma - \beta}{3}. \)

Acknowledgments. The work of N.H. is partially supported by JSPS KAKENHI Grant Numbers JP25220702, JP15H03630. The work of C.L. is partially supported by NNSFC Grant No.11461074. The work of P.I.N. is partially supported by CONACYT and PAPIIT project IN100616.

References

Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka 560-0043, Japan
E-mail address: nhayashi@math.sci.osaka-u.ac.jp