THE GROWTH OF THE VORTICITY GRADIENT FOR THE TWO-DIMENSIONAL EULER FLOWS ON NONSMOOTH DOMAINS

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Let Ω be a two-dimensional domain. We are concerned with the Euler equations on Ω in the vorticity formulation:

$$\omega_t + (u \cdot \nabla)\omega = 0, \quad \omega(x,0) = \omega_0(x). \tag{1}$$

Here ω is the fluid vorticity, and u is the velocity of the flow determined by the Biot-Savart law. We impose the no flow condition for the velocity at the boundary: $u \cdot n = 0$ on $\partial\Omega$, where n is the unit normal vector on the boundary. This implies the formula:

$$u(x,t) = \nabla^{\perp} \int_{\Omega} G_{\Omega}(x,y)\omega(y,t)dy,$$

where G_{Ω} is the Green function for the Dirichlet problem in Ω and $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$. The movement of a fluid particle, placed at a point $X \in \Omega$, is defined as the solution of the Cauchy problem

$$\frac{d\gamma_X(t)}{dt} = u(\gamma_X(t),t), \quad \gamma_X(0) = X,$$

and the vorticity ω is advected by

$$\omega(x,t) = \omega_0(\gamma_x^{-1}(t)).$$

Global regular solutions to the Euler equatins (1) in smooth bounded domains were proved by Wolibner [9] and Hölder [2] and there are huge literature on this problem. Recently, there are growing interests in the study of (1) in nonsmooth domains. Existence of global weak solutions, with $u \in L^{\infty}(\mathbb{R}_+; L^2(\Omega))$ and $\omega \in L^{\infty}(\mathbb{R}_+ \times \Omega)$, was proved by Taylor [8] for convex domains and by Gèrard-Varet and Lacave [1] for more general (possibly not convex) domains. Uniqueness of the solution to the Euler equations (1) on domains with corners was shown by Lacave, Miot and Wang [7] for acute angles. For obtuse corners, Lacave [6] proved uniqueness of the solution under the assumption that the support of the vorticity never intersects the boundary. We are concerned with the question how fast the maximum of the gradient of the vorticity can grow as $t \to \infty$. When Ω is a smooth bounded domain, the best known upper bound on the growth is double-exponential [11], while the question whether such upper bound is sharp had been open for a long time. In 2014, Kiselev and Sverak [4] answered the question affirmatively for the case Ω is a disk. They gave an example of the solution growing with double exponential rate. For a general domain with C^3-boundary see [10]. On the other hand, Kiselev and Zlatos [5] considered the 2D Euler flows on some bounded domain with certain cusps. They showed that the gradient of vorticity blows up at the cusps in finite time. These solutions are constructed by imposing certain symmetries on the initial data, which leads to a *hyperbolic flow scenario* near a stagnation point on the boundary. More precisely, by the hyperbolic flow scenario, particles on the boundary (near the stagnation point) head for the stagnation point for all time. Moreover

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the relation between this scenario and the geometry of the boundary plays a crucial role in the double exponential growth or the formation of the singularity. Thus it would be an interesting question to ask how the geometry of the boundary affects the growth of the solution. In [3], Miura, Yoneda and the author considered the Euler equations (1) on the unit square and under a simple symmetry condition the growth of the Lipschitz constant of the vorticity on the boundary is shown to be at most single exponential at the stagnation point. In this talk, we are concerned with more general cases; the growth of the Lipschitz norm of the vorticity in bounded domains with general corners or cusps.

Definition 1. (i) Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain $0 < \theta < 2\pi$ with $\theta \neq \pi$. We say that $\partial\Omega$ has a corner of angle θ $(0 < \theta < 2\pi)$ at $\xi \in \partial\Omega$, if there exist constants $r_0 > 0$ and $0 \leq \theta_0 < 2\pi$ such that, $\Omega \cap B(\xi, r_0) = \{x = (x_1, x_2) : \theta_0 < \arg(x - \xi) < \theta_0 + \theta\} \cap B(\xi, r_0)$.
(ii) Let Ω be a domain with corners given in (i). We say Ω is symmetric with respect to the corner if $\theta_0 = -\frac{\theta}{2}$ and Ω is symmetric along the x_1-axis.

Without loss of generality, by translation, rotation and scaling, we may assume that

$$\begin{cases} \text{diam}(\Omega) < 1 \text{ and } 0 \in \partial\Omega, \\ \partial\Omega \text{ has a corner of angle } \theta \text{ at } 0 \text{ with } \theta_0 = 0 \text{ in Definition 1.} \end{cases} \tag{2}$$

We now focus on the growth of the Lipschitz constant of ω with $0 \in \partial\Omega$

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|}.$$

Our first result concerns the domain with the corner with the angle $\theta \leq \pi/2$.

Theorem 2. *Let Ω be a simple connected domain satisfying (2) and ω_0 be a Lipschitz function.*
(a) For $0 < \theta < \frac{\pi}{2}$, there exists a constant $C > 0$ depending only on Ω such that

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|} \leq \|\omega_0\|_{\text{Lip}} e^{C\|\omega_0\|_\infty t} \quad \textit{for } t > 0.$$

Moreover there exist an initial data ω_0 and a constant $C > 0$ such that

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|} \geq Ce^{Ct} \quad \textit{for } t > 0.$$

(b) For $\theta = \frac{\pi}{2}$, there exists an initial data ω_0 with $\|\omega_0\|_{\text{Lip}} > 1$ such that

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|} \geq \|\omega_0\|_{\text{Lip}}^{C\exp(Ct)} \quad \textit{for } t > 0.$$

We next consider the case $\theta > \pi/2$. In this case, we will see that the vorticity can lose continuity instantaneously.

Theorem 3. *Let Ω be a simply connected bounded domain satisfying (2). If $\pi/2 < \theta < \pi$, there are an initial data $\omega_0 \in C(\overline{\Omega})$ and its solution ω such that $\omega(t)$ instantaneously loses continuity in space. Furthermore, if $\pi < \theta < 2\pi$ and Ω is symmetric with respect to the corner, there also exist $\omega_0 \in C(\overline{\Omega})$ and its solution ω such that $\omega(t)$ instantaneously loses continuity.*

Next we consider domains with cusps. Let

$$\Omega_1 = \{(x_1, x_2) : x_2 > 0, (x_1 - 1/2)^2 + x_2^2 < 1/4, x_1^2 + (x_2 - 1/2)^2 > 1/4\}.$$

Note that Ω_1 has an outward-pointing cusp at 0.

Theorem 4. *Let* ω_0 *be a Lipschitz function on* Ω_1*. Consider the Euler equations* (1) *on* Ω_1*. There exists a constant* $C > 0$ *such that*

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|} \le \|\omega_0\|_{\mathrm{Lip}}(1 + C\|\omega_0\|_\infty t) \quad \textit{for } t > 0.$$

Moreover there exist an initial data ω_0 *and a constant* $C > 0$ *such that*

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|} \ge 1 + t \quad \textit{for } t > 0.$$

Theorem 5. *There exists a domain* Ω *with an outward-pointing cusp at* 0 *such that the following statement holds: Let* ω_0 *be a Lipschitz function on* Ω*. There exists a constant* $C > 0$ *such that*

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|} \le \|\omega_0\|_{\mathrm{Lip}}(1 + C\log(1 + \|\omega_0\|_\infty t)) \quad \textit{for } t > 0.$$

Moreover there exist an initial data ω_0 *and a constant* $C > 0$ *such that*

$$\sup_{x\in\Omega} \frac{|\omega(x,t) - \omega(0,t)|}{|x|} \ge 1 + \log(1 + t) \quad \textit{for } t > 0.$$

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