New measure-valued solutions for Euler system

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As the recent results (e.g., [2], [1]) show, even entropy admissible weak solutions to the Euler equations are not unique. Motivated by a discussion on the perfect gas case, we introduce a new concept of a dissipative measure-valued solution to the Euler system

$$
\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) &= 0, && (1)\\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u}\otimes \mathbf{u}) + \nabla_x p(\varrho,\vartheta) &= 0, && (2)\\
\partial_t \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + p(\varrho,\vartheta)\right)\mathbf{u}\right] &= 0, && (3)\\
\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) &\geq 0, && (4)
\end{aligned}
$$

where the pressure $p = p(\varrho,\vartheta)$, the specific internal energy $e = e(\varrho,\vartheta)$, and the specific entropy $s = s(\varrho,\vartheta)$ are interrelated through *Gibbs' equation.* Here $\varrho = \varrho(t,x)$ is the mass density, $\mathbf{u} = \mathbf{u}(t,x)$ is the velocity, and $\vartheta = \vartheta(t,x)$ is the (absolute) temperature of a compressible inviscid fluid.

Our goal is to address the problem of weak (measure-valued)strong uniqueness for the Euler system (1–4). Accordingly, we focus on identifying the largest class possible of measurevalued solutions, in which such a result holds, rather than the optimal one with respect to the expected regularity of solutions.

One of the principal difficulties is the hypothetical presence of vacuum zones on which the underlying equations fail to provide any control on the behavior of solutions. To avoid this problem, new phase variables must be considered - the density ϱ, the internal energy density $E = \varrho e$, and the momentum $\mathbf{m} = \varrho \mathbf{u}$.

Definition 1. [Dissipative measure-valued solution]
A family of probability measures $\{Y_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$*,*

$$(t,x) \mapsto Y_{t,x} \in L^\infty_{\text{weak}-(*)}((0,T)\times\Omega;\mathcal{P}(\mathcal{F})),$$

and the dissipation defect $\mathcal{D} \in L^\infty(0,T)$ *represent a* dissipative measure-valued solution *of the Euler system (1–4) with the initial data* $Y_{0,x}$ *if:*

$$\left[\int_\Omega \langle Y_{t,x};\varrho\rangle \varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\langle Y_{t,x};\varrho\rangle \partial_t\varphi + \langle Y_{t,x};\mathbf{m}\rangle\cdot\nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t,$$

for a.a. $\tau \in (0,T)$ *and for any* $\varphi \in C^1([0,T]\times\Omega)$*;*

$$
\begin{aligned}
&\left[\int_\Omega \langle Y_{t,x};\mathbf{m}\rangle\cdot\boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau}\\
&= \int_0^\tau\int_\Omega \left[\langle Y_{t,x};\mathbf{m}\rangle\cdot\partial_t\boldsymbol{\varphi} + \left\langle Y_{t,x};\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right\rangle : \nabla_x\boldsymbol{\varphi} + \langle Y_{t,x};p(\varrho,E)\rangle\,\mathrm{div}_x\boldsymbol{\varphi}\right] \ \mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^\tau \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mu_{\mathcal{R}},
\end{aligned}
$$

for a.a. $\tau \in (0,T)$ *and for any* $\boldsymbol{\varphi} \in C^1([0,T] \times \Omega; R^3)$*;*

$$\left[\int_\Omega \langle Y_{t,x}; \varrho Z\left(s(\rho,E)\right) \rangle \varphi \,\mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$\geq \int_0^\tau \int_\Omega \Big[\langle Y_{t,x}; \varrho Z\left(s(\rho,E)\right) \rangle \partial_t \varphi + \langle Y_{t,x}; Z\left(s(\rho,E)\right) \mathbf{m} \rangle \cdot \nabla_x \varphi \Big] \,\mathrm{d}x \,\mathrm{d}t,$$

for a.a. $\tau \in (0,T)$*, any* $\varphi \in C^1([0,T] \times \Omega)$*,* $\varphi \geq 0$*,* $Z \in BC(R)$*,* $Z' \geq 0$*;*

$$\left[\int_\Omega \left\langle Y_{t,x}; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + E \right\rangle \,\mathrm{d}x \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) = 0,$$

where the dissipation defect $\mathcal{D}$ *dominates the signed measure*

$$\mu_{\mathcal{R}} \in \mathcal{M}([0,T] \times \Omega; R^{3\times 3}),$$

specifically,

$$\|\mu_{\mathcal{R}}\|_{\mathcal{M}([0,\tau)\times\Omega; R^{3\times3})} \leq c \int_0^\tau \mathcal{D}(t) \,\mathrm{d}t,$$

for a.a. $\tau \in (0,T)$*.*

Our main result is as follows.

Theorem 1. [Weak (measure-valued) - strong uniqueness principle] *Let the thermodynamic functions* $e = e(\varrho,\vartheta)$*,* $s = s(\varrho,\vartheta)$*, and* $p = p(\varrho,\vartheta)$ *satisfy Gibbs' relation, the hypothesis of thermodynamic stability, and let*

$$|p(\varrho,\vartheta)| \leq c(1 + \varrho + \varrho|s(\varrho,\vartheta)| + \varrho e(\varrho,\vartheta)).$$

Let $[r,\Theta,\mathbf{U}]$ *be a continuously differentiable classical solution of the Euler system (1–3) in* $(0,T) \times \Omega$ *starting from the initial data* $(r_0, \Theta_0, \mathbf{U}_0)$ *satisfying*

$$r_0 > 0, \;\; \Theta_0 > 0.$$

Assume that $[Y_{t,x}; \mathcal{D}]$ *is a dissipative measure valued solution of the same problem in the sense specified in Definition 1 such that*

$$Y_{0,x} = \delta_{[r_0(x), r_0 e(r_0,\Theta_0)(x), r_0 \mathbf{U}_0(x)]} \text{ for a.a. } x \in \Omega.$$

Then $\mathcal{D} = 0$ *and*

$$Y_{t,x} = \delta_{[r(t,x), re(r,\Theta)(t,x), r\mathbf{U}(t,x)]} \text{ for any } (t,x) \in (0,T) \times \Omega.$$

References

[1] E. Chiodaroli, C. De Lellis, and O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. *Comm. Pure Appl. Math.*, **68**(7):1157–1190, 2015.

[2] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, **195**(1):225–260, 2010.