Partial smoothing effect and energy-dissipation for an Allen-Cahn equation with non-decreasing constraint

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This talk is based on a joint work with Messoud Efendiev (München).

1 Introduction

This talk is concerned with an Allen-Cahn equation with non-decreasing constraint,

$$\text{(irAC)} \qquad u_t = \Big(\Delta u - W'(u)\Big)_+ \ \text{ in } \ \Omega \times (0,\infty),$$

where $W'(u) = u^3 - \kappa u$ (with $\kappa > 0$) is the derivative of a double-well potential $W(u)$ simply given by

$$W(u) := \frac{1}{4}u^4 - \frac{\kappa}{2}u^2$$

and where $(\,\cdot\,)_+$ stands for the positive-part function and Ω is a smooth bounded domain of $\mathbb{R}^N$. Equation (irAC) is a constrained (more precisely, a *strongly irreversible*) variant of the celebrated Allen-Cahn equation,

$$\text{(AC)} \qquad u_t = \Delta u - W'(u) \ \text{ in } \ \Omega \times (0,\infty),$$

which has been well studied and is known for a phase-separation model. Moreover, (irAC) also appears in a special setting of a phase field model of crack-propagation, where $u(x,t)$ stands for the phase parameter describing the degree of damage and therefore it is supposed to be non-decreasing. Furthermore, (AC) and (irAC) share a common Lyapunov energy functional,

$$\mathcal{F}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega W(u(x))\,\mathrm{d}x$$

for $u \in H^1_0(\Omega) \cap L^4(\Omega)$. Namely, $t \mapsto \mathcal{F}(u(t))$ is non-increasing along the evolution of each solution $u(t)$ to (AC) and (irAC). Here we remark that (AC) can be formulated as an L^2-gradient flow of $\mathcal{F}$, that is,

$$u_t(t) = -\mathcal{F}'(u(t)) \ \text{ in } L^2(\Omega), \quad 0 < t < \infty.$$

On the other hand, (irAC) is rewritten as a *constrained* gradient flow,

$$u_t(t) = \Big(-\mathcal{F}'(u(t))\Big)_+ \ \text{ in } L^2(\Omega), \quad 0 < t < \infty,$$

which is not in a divergence form and often classified as a fully nonlinear PDE.

Due to the presence of the positive-part function in (irAC), $u(x,t)$ is non-decreasing in time for a.e. $x \in \Omega$. In particular, if $u_0 \geq 0$ a.e. in Ω, then $u(x,t) \geq u_0(x)$ for a.e. $x \in \Omega$ and $t > 0$, which also implies

$$\|u(t)\|_{L^p(\Omega)} \geq \|u_0\|_{L^p(\Omega)} \quad \text{for all } \ t > 0 \ \text{ and } \ p \in [1,\infty].$$

Hence there exists no absorbing set in any L^p spaces, and therefore, no global attractor exists in $L^p(\Omega)$. This fact exhibits a clear contrast to the classical Allen-Cahn equation (AC), which always admits a global attractor in $L^p(\Omega)$. Moreover, it can be also regarded as a conclusion of a lack of full energy-dissipation for (irAC). On the other hand, $u(x,t)$ never evolves at $x \in \Omega$ where $\Delta u(x,t) - W'(u(x,t))$ is negative. Therefore, one cannot expect smoothing-effect for (irAC) in a usual sense. The main purpose of this talk is to exhibit how one can find out *partial* energy-dissipation and smoothing effect for (irAC).

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2 Main results

Let us consider the Cauchy-Dirichlet problem (P) for (irAC),

$$u_t = \Big(\Delta u - W'(u)\Big)_+ \quad \text{in} \quad \Omega \times (0,\infty),$$
$$u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0.$$

Here and henceforth, we are concerned with L^2 solutions of (irAC), that is,

Definition (Solution to (irAC))

For $T > 0$, a function $u \in C([0,T]; L^2(\Omega))$ is called a *solution of* (irAC) on $[0,T]$, if the following conditions are all satisfied:

(i) $u \in W^{1,2}(\delta, T; L^2(\Omega)),\ C([\delta,T]; H_0^1(\Omega) \cap L^4(\Omega)) \cap L^2(\delta,T;H^2(\Omega)) \cap L^6(\delta,T;L^6(\Omega))$ for any $0 < \delta < T < \infty$,

(ii) it holds that

$$u_t = \Big(\Delta u - W'(u)\Big)_+ \quad \text{for a.e. } (x,t) \in \Omega \times (0,\infty). \tag{1}$$

Our main results read,

Theorem 1 (Existence and partial smoothing effect [1])

We set

$$D_r := \Big\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega) \colon \|(\Delta u - W'(u))_-\|_2^2 \le r \Big\}$$

for each $r > 0$. Then

(i) Let u_0 belong to the closure $\overline{D_r}^{L^2}$ of D_r in $L^2(\Omega)$. Then (P) admits a solution $u = u(x,t)$ on $[0,T]$ satisfying

$$u \in L^2(0,T; H_0^1(\Omega)) \cap L^4(0,T;L^4(\Omega)),$$
$$u(t) \in D_r \quad \text{for all} \quad t \in (0,T]$$

for any $0 < T < \infty$.

(ii) If u_0 belongs to the closure $\overline{D_r}^{H_0^1 \cap L^4}$ of D_r in $H_0^1(\Omega) \cap L^4(\Omega)$, then it further holds that

$$u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^6(0,T;L^6(\Omega)),$$
$$u \in C([0,T]; H_0^1(\Omega) \cap L^4(\Omega))$$

for any $0 < T < \infty$.

(iii) If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^6(\Omega)$, then $u \in C_w([0,T]; H^2(\Omega) \cap L^6(\Omega))$ and $u_t \in L^2(0,T;H_0^1(\Omega))$ for any $0 < T < \infty$.

Theorem 2 (Partial energy-dissipation [1])

Fix $r > 0$ and denote by $u(t) = u(\cdot, t)$ the solution to (P) constructed in Theorem 1. Then there exist constants C_r and C such that the following (i)–(iii) hold true:

(i) For any $u_0 \in \overline{D_r}^{H^1_0 \cap L^4}$, it holds that

$$\phi(t) \le C_r + e^{-2\kappa t}\phi(0), \quad \|\Delta u(t) - W'(u(t))\|_2^2 \le C_r + \frac{C(\phi(0)+1)}{t},$$

where $\phi(t) := (1/2)\|\nabla u(t)\|_2^2 + (1/4)\|u(t)\|_4^4$, for all $t \ge 0$.

(ii) Fix $u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \cap L^6(\Omega)$. For any $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that

$$\|\Delta u(t) - W'(u(t))\|_2^2 \le \|(\Delta u_0 - W'(u_0))_-\|_2^2 + \varepsilon$$

for all $t \ge \tau_\varepsilon$.

Theorem 3 (Reformulation of (irAC) as an obstacle problem [1])

For any $u_0 \in \overline{D_r}^{L^2}$, (P) admits a solution $u = u(x,t)$ which also solves

$$\text{(OP)} \qquad \begin{cases} u \ge u_0, \quad u_t \ge \Delta u - W'(u) & \text{in } \Omega \times (0,\infty), \\ (u - u_0)\,(u_t - \Delta u + W'(u)) = 0 & \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \end{cases}$$

Moreover, such a solution to (P) as well as (OP) is uniquely determined by $u_0 \in \overline{D_r}^{L^2}$.

Equation (irAC) is classified as a fully nonlinear parabolic equation, which is formulated in a general form $u_t = F(D^2 u)$ with a nonlinear function F and the Hessian matrix $D^2 u$. Here we shall reformulate the equation as a generalized gradient flow (of subdifferential type), which is fitter to distributional frameworks and energy techniques. By applying the (multivalued) inverse mapping $\alpha(\,\cdot\,)$ of $(\,\cdot\,)_+$ to both sides, (irAC) is reduced to

$$\text{(irAC)}_{\text{DN}} \qquad \alpha(u_t) \ni \Delta u - W'(u) \quad \text{in} \quad \Omega \times (0,\infty).$$

The inverse mapping α of $(\,\cdot\,)_+$ can be decomposed as follows:

$$\alpha(s) = s + \partial I_{[0,\infty)}(s), \quad \partial I_{[0,\infty)}(s) = \begin{cases} 0 & \text{if } \ s > 0 \\ (-\infty, 0] & \text{if } \ s = 0 \\ \emptyset & \text{if } \ s < 0 \end{cases} \quad \text{for } \ s \in \mathbb{R}, \tag{2}$$

where $\partial I_{[0,\infty)}$ stands for the subdifferential of the indicator function $I_{[0,\infty)}$ over the half-line $[0, +\infty)$. The Cauchy-Dirichlet problem (P) for (irAC) is equivalently given as

$$u_t + \eta - \Delta u + W'(u) = 0, \quad \eta \in \partial I_{[0,\infty)}(u_t) \quad \text{in } \ \Omega \times (0,\infty), \tag{3}$$

$$u = 0 \quad \text{on } \ \partial\Omega \times (0,\infty), \tag{4}$$

$$u = u_0 \quad \text{in } \ \Omega. \tag{5}$$

Furthermore, comparing (3) with (irAC), one can immediately find the relation,

$$\eta = -\Big(\Delta u - W'(u)\Big)_-, \tag{6}$$

where $(\,\cdot\,)_-$ stands for the negative part function, i.e., $(s)_- := \max\{-s, 0\} \ge 0$.

References

[1] Akagi, G. and Efendiev, M., Allen-Cahn equation with strong irreversibility, preprint.