Long time oscillation of solutions of nonlinear Schrödinger equations near minimal mass ground state

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1 Introduction

This talk is based on the joint work [9] with S. Cuccagna (Trieste University).

In this talk we consider a nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\partial_t u = -\Delta u + g(|u|^2)u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1.1}$$

with $u(0) = u_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C}) := \{u \in H^1(\mathbb{R}^3, \mathbb{C}) \mid u \text{ is radially symmetric}\}$ and $g \in C^\infty([0,\infty), \mathbb{R})$ with $g(0) = 0$ and $|g^{(n)}(s)| \le C_n s^{p-n}$ in $s \ge 1$ for some $p < 2$ and $C_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

We assume the existence of a one parameter family of ground states, i.e. there exists an open interval $\mathcal{O} \subset (0,\infty)$ and a C^2 map

$$\mathcal{O} \ni \omega \mapsto \phi_\omega \in H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{R}), \tag{1.2}$$

s.t. ϕ_ω are positive and solve

$$0 = -\Delta\phi_\omega + \omega\phi_\omega + g(\phi_\omega^2)\phi_\omega. \tag{1.3}$$

Notice that $e^{\mathrm{i}\omega t}\phi_\omega$ are solutions of (1.1), which are also called ground states. We further assume that the map $\omega \mapsto \|\phi_\omega\|_{L^2}$ has a nondegenerate local minimum, i.e.

$$\exists \omega_* \in \mathcal{O} \text{ s.t. } q'(\omega_*) = 0 \text{ and } q''(\omega_*) > 0, \text{ where } q(\omega) := \frac{1}{2}\|\phi_\omega\|^2_{L^2(\mathbb{R}^3,\mathbb{C})}. \tag{1.4}$$

Remark 1.1. The above hypotheses have been numerically verified for various equations involving *saturated* nonlinearities such as $g(s) = \frac{s}{1+s}$, relevant in optics. Other examples are in Buslaev–Grikurov [2]. For an analytical result for double power nonlinearity in one dimension, see [12].

By (1.4), we have $q'(\omega) < 0$ for $\omega \in (\omega_* - \delta, \omega_*)$ and $q'(\omega) > 0$ for $\omega \in (\omega_*, \omega_* + \delta)$ for some $\delta > 0$. It is well known that under standard nondegeneracy assumption, the ground state $e^{\mathrm{i}\omega t}\phi_\omega$ is orbitally stable (resp. unstable) if $q'(\omega) > 0$ ($q'(\omega) < 0$), see e.g. [10]. Therefore, ω_* is the critical frequency dividing the stable and unstable ground states. Comech–Pelinovsky [4] proved that this critical ground state $e^{\mathrm{i}\omega_* t}\phi_{\omega_*}$ is orbitally unstable.

Marzuola *et al.* [13] considered the framework of Comech–Pelinovsky [4] and developed a systematic numerical exploration of solutions of both the finite dimensional approximation and of the full NLS. While observing the patterns of Comech–Pelinovsky [4], Marzuola *et al.* [13] identified another class of initial data near the minimal mass ground state $e^{\mathrm{i}\omega_* t}\phi_{\omega_*}$. The corresponding solutions look like $e^{\mathrm{i}\vartheta(t)}\phi_{\omega(t)}$ locally in space with $\omega(t)$ displaying an oscillating motion, which appears periodic. A similar pattern had previously been observed also by Buslaev–Grikurov [2, fig. 2 case $\alpha = 0.5$].

Our aim is to rigorously justify the observation of Marzuola *et al.* [13] and provide a theoretical explanation of the oscillation phenomena. Very roughly, our main result is the following.

Theorem 1.2 (Cuccagna-M. [9]). *For any $M \in \mathbb{N}$, $M \geq 2$, there exists an open set $\mathcal{U}_M \subset H^1_{\mathrm{rad}}$ near ϕ_{ω_*} contained in $\{u \in H^1_{\mathrm{rad}} \mid Q(u) > Q(\phi_{\omega_*})\}$ s.t. the solution of (NLS) with initial datum $u_0 \in \mathcal{U}_M$ can be expressed as*

$$u(t) = e^{\mathrm{i}\theta(t)}\phi_{\omega(t)} + O(\epsilon^{3/2}), \quad t \in [0, T]$$

where $\epsilon := \sqrt{Q(u) - Q(\phi_{\omega_*})}$ and $T = \epsilon^{-M}$. Moreover, if we set $\omega = \omega_+ + \epsilon\zeta$, where ω_+ is the solution of $Q(\phi_\omega) = Q(u)$ with $\omega_+ > \omega_*$, we have

$$\frac{d^2}{dt^2}\zeta = -A^{-1}\left(\sqrt{2q''(\omega_*)}\zeta + \frac{1}{2}q''(\omega_*)\zeta^2\right) + O(\epsilon).$$

Here, $Q(u) := \frac{1}{2}\|u\|^2_{L^2}$

Remark 1.3. The open sets $\mathcal{U}_M \subset H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C})$ are not neighborhoods of ϕ_{ω_*}.

We now explain the strategy of the proof. For problems related to the dynamics near ground states, following the classical works such as [19, 16], it is quite natural to introduce coordinates which are in part discrete and in part continuous coordinates and which are related to the spectral decomposition of the linearization of (1.1). For example, when we study the asymptotic stability of ground states (i.e. showing solutions near ground states decompose to ground state and linear wave) for the case $q'(\omega) > 0$, we can decompose the solution $u(t)$ as

$$u(t) = e^{\mathrm{i}\vartheta(t)}(\phi_{\omega(t)} + r(t)).$$

Here, $r(t)$ is symplectic orthogonal to the kernel of the linearized operator. If we assume that there are no internal modes (non-zero discrete spectrum) for the linearized operator, then (ϑ, ω) can be viewed as the discrete coordinates and r as the continuous coordinate, with (ϑ, ω, r) a complete system of independent coordinates. Therefore, the dynamics of $u(t)$ reduces to the dynamics of $\vartheta(t)$, $\omega(t)$ and $r(t)$. As in [7, 1] it is natural to replace the coordinate system (ϑ, ω, r) with (ϑ, Q, r), where $Q = \frac{1}{2}\|u\|^2_{L^2}$ is a first integral of motion, and by an elementary application of Noether's principle, to decrease the number of coordinates, reducing to a NLS–like equation on r. Then the proof of asymptotic stability results from the proof of the scattering of r.

The above argument, is based on the fact that the generalized kernel of the linearized operator is 2 dimensional, which is a consequence of $q'(\omega) \neq 0$. In the case $q'(\omega) = 0$ the generalized kernel becomes 4 dimensional, and instead of the previous ansatz with coordinates (ϑ, ω, r), we have

$$u(t) = e^{\mathrm{i}\vartheta(t)}\left(\phi_{\omega(t)} + \lambda(t)\psi_3 + \mu(t)\psi_4 + r(t)\right),$$

where ψ_3 and ψ_4 are additional generalized eigenfunctions. Thus, under the assumption that there exist no internal modes, our problem reduces to the study of the dynamics of $(\vartheta, \omega, \lambda, \mu, r)$. Unlike in the case $q'(\omega) \neq 0$, we cannot replace ω by Q. But we can replace μ by Q. Then, by an elementary application of Noether's principle, we are left with (ω, λ, r). The equations for modulation parameters (ω, λ) have already been studied in the literature, see (4.11) in [4] or (3.5) in [13], but are not well understood. Here we add to their understanding by following an approach initiated in [6]. That is, using the Hamilton structure of NLS (1.1), we move to a Darboux system of coordinates. In Darboux coordinates we have

$$\dot{\omega} = A^{-1}\partial_\lambda E, \quad \dot{\lambda} = -A^{-1}\partial_\omega E$$

where E is the energy (Hamiltonian) of the NLS and $A = A(\omega) > 0$ is a function that here we can assume constant. If we expand $E = E_f(Q, \omega, \lambda, 0) +$ "terms with r", where E_f is the energy of the finite dimensional part, and think the second term as an error, we obtain

$$\dot{\omega} = A^{-1}\partial_\lambda E_f(Q, \omega, \lambda, 0) + \text{error}, \quad \dot{\lambda} = -A^{-1}\partial_\omega E_f(Q, \omega, \lambda, 0) + \text{error}.$$

Ignoring errors, this is a 2 dimensional Hamiltonian system with energy $E_f \sim 2^{-1}A\lambda^2 + d(\omega) - \omega Q$, where $d(\omega) = E(\phi_\omega) + \omega q(\omega)$ satisfies $d'(\omega) = q(\omega)$. So, thinking $2^{-1}A\lambda^2$ as the kinetic energy and $V_Q(\omega) = d(\omega) - \omega Q$ as potential energy, we see that ω is approximately the position function in Newton's equation with the potential well V_Q, with λ the momentum. If $Q > q(\omega_*)$, the well has a local minimum at a point $\omega = \omega_+ > \omega_*$. Thus, in this case, even starting from $\omega(0) = \omega_*$, ω will oscillate around ω_+, consistently with the numerical observations by Marzuola *et al.* [13, Fig. 4–6]. Similarly, we see that if $Q < q(\omega_*)$, then $V(Q, \omega)$ is monotically increasing and ω will fall to the unstable side ($\omega < \omega_*$) without oscillation.

While we prove oscillations for very long times, the question on what happens asymptotically to these oscillating patterns remains open. In [13, Conjecture 1.1] it is suggested that asymptotically these oscillating solutions should scatter to a stable ground state. Reference is made to possible radiation damping phenomena and to the damped oscillations observed numerically in some cases for the mass critical saturated NLS by LeMesurier *et al.* [11, figures 7 and 16]. For further comments and references see also the discussion in [18, Sect. 9.3.2–9.3.3].

We think that it is plausible that a radiation damping phenomenon like in [17, 3, 6, 8] will prove Conjecture 1.1 in [13]. However, it is almost certain that the type of coordinates used in the present work, which originate from the analysis in Comech and Pelinovsky [4], are inadequate to prove the conjecture. There should exist more "nonlinear" coordinates, possibly related to the ones used in the study of the blow up by Perelman [15] and Merle and Raphael [14].

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