On differential Harnack inequality on Riemannian manifolds and Ricci flow

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Differential Harnack inequality (DHI) is an important tool in the study of geometric analysis and PDEs. The aim of this talk is to present some new results in the study of DHI on complete Riemannian manifolds with the so-called $CD(K,m)$-condition and on geometric flows, in particular, on Ricci flow and backward Ricci flow.

Let M be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below by a non-positive constant $-K$, i.e., $Ric \geq -K$, where $K \geq 0$ is a constant. In their famous paper [5], P. Li and S.-T. Yau proved that for any positive solution to the heat equation

$$\partial_t u = \Delta u, \tag{1}$$

the following DHI holds

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{\sqrt{2}(\alpha - 1)}. \tag{2}$$

In particular, when $Ric \geq 0$, taking $\alpha \to 1$, it holds

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}. \tag{3}$$

This inequality is sharp as it becomes equality when $M = \mathbb{R}^n$ and $u(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x\|^2}{4t}}$.

As a consequence of the Li-Yau differential Harnack inequality (2), Li and Yau obtained the following parabolic Harnack inequality (PHI) for any positive solution to the heat equation (1): for any $x, y \in M$ and $0 < s < t$,

$$\frac{u(x,s)}{u(y,t)} \leq \left(\frac{t}{s}\right)^{n\alpha/2} \exp\left(\frac{\alpha^2 d^2(x,y)}{4(t-s)} + \frac{n\alpha K}{2(\alpha-1)}(t-s)\right). \tag{4}$$

In particular, when $Ric \geq 0$, it holds

$$\frac{u(x,s)}{u(y,t)} \leq \left(\frac{t}{s}\right)^{n/2} \exp\left(\frac{d^2(x,y)}{4(t-s)}\right).$$

Moreover, when $Ric \geq 0$, it has been shown by Li and Yau that the fundmental solution to the heat equation (1) satisfies the two-sides Gaussian estimates: for any $\varepsilon > 0$, there exists constants $C_{1,\varepsilon,n} > 0$ and $C_{2,\varepsilon,n} > 0$ which depend only on ε and n such that for any $x, y \in M$ and $t > 0$

$$\frac{C_{1,\varepsilon,n}}{V(B(x,\sqrt{t}))} e^{-\frac{d^2(x,y)}{4(1-\varepsilon)t}} \leq p_t(x,y) \leq \frac{C_{2,\varepsilon,n}}{V(B(x,\sqrt{t}))} e^{-\frac{d^2(x,y)}{4(1+\varepsilon)t}}. \tag{5}$$

In [3], R. Hamilton proved a dimension free differential Harnack inequality. More precisely, let M be a compact Riemannian manifold with $Ric \geq -K$, where $K \geq 0$ is a constant, then for any positive and bounded solution of (1), it holds

$$\frac{|\nabla u|^2}{u^2} \leq \left(2K + \frac{1}{t}\right) \log(A/u), \tag{6}$$

where $A := \sup\{u(t,x) : x \in M, t \geq 0\}$. Indeed, such inequality also holds when M is a complete Riemannian manifold with $Ric \geq -K$. Moreover, on compact Riemannian manifolds with $Ric \geq -K$, Hamilton also proved the following Li-Yau type differential Harnack inequality for any positive solution to the heat equation (1)

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{\partial_t u}{u} \leq \frac{n}{2t}e^{4Kt}. \tag{7}$$

In particular, when $K = 0$, (7) reduces to (3). We call (2) the Li-Yau differential Harnack inequality, and (7) the Li-Yau-Hamilton differential Harnack inequality. As a consequence of (7), Hamilton proved that for any $x, y \in M$ and $0 < s < t$, the following parabolic Harnack inequality holds

$$\frac{u(x,s)}{u(y,t)} \leq \left(\frac{t}{s}\right)^{n/2} \exp\left(\frac{e^{2Ks}[1+2K(t-s)]}{4}\frac{d^2(x,y)}{t-s} + \frac{n}{2}[e^{2Kt} - e^{2Ks}]\right). \tag{8}$$

In [2], R. Hamilton introduced the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric. \tag{9}$$

He raised the program to use the Ricci flow to prove the Poincaré conjecture. In [3, 4], Hamilton used the differential Harnack inequalities associated to the solutions to the backward heat equation to prove some monotonicity formulas for parabolic flows. In [17], G. Perelman gave a gradient formulation to the Ricci flow, and proved the monotonicity of the $\mathcal{F}$-entropy and the $\mathcal{W}$-entropy along the conjugate heat equation associated to the Ricci flow. This leads the no local collapsing theorem and final resolution of the Poincaré conjecture and Thurston's geometrization conjecture.

Let (M, g) be an n-dimensional complete Riemannian manifold, $\phi \in C^2(M)$ be a potential function, and $d\mu = e^{-\phi}dv$, where v is the volume element on (M, g). The weighted Laplacian on (M, g, ϕ), called also the Witten Laplacian, is defined by

$$L = \Delta - \nabla\phi \cdot \nabla.$$

For any $u, v \in C_0^\infty(M)$, the integration by parts formula holds

$$\int_M \langle \nabla u, \nabla v\rangle d\mu = -\int_M Luv d\mu = -\int_M uLv d\mu.$$

By [1], for any $u \in C^\infty(M)$, the generalized Bochner formula holds

$$L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu\rangle = 2|\nabla^2 u|^2 + 2Ric(L)(\nabla u, \nabla u), \tag{10}$$

where $\nabla^2 u$ is the Hessian of u, $|\nabla^2 u|$ denotes its Hilbert-Schmidt norm, and

$$Ric(L) = Ric + \nabla^2\phi.$$

In the literature, $Ric(L)$ is called the Bakry-Emery Ricci curvature associated with the Witten Laplacian L on (M, g, ϕ).

For any $m \in [n, \infty]$, the m-dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian L on (M, g, ϕ) is defined as follows

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m - n}.$$

We now make a convention: when $m = n$, ϕ is a constant, and when $m = \infty$, $Ric_{\infty,n}(L) = Ric(L)$.

Following Bakry and Emery, for any constant $m \in [n, \infty]$ and $K \in \mathbb{R}$, we call that (M, g, ϕ) or L satisfies the $CD(K, m)$ (curvature-dimension) condition, if and only if

$$Ric_{m,n}(L) \geq K.$$

In the case $m \in \mathbb{N} \cap [n, \infty) Ric_{m,n}(L)$ has the following geometric interpretation. Define the warped product metric on $M^n \times S^{m-n}$ by

$$\widetilde{g} = g_M \bigoplus e^{-\frac{2\phi}{m-n}} g_{S^{m-n}}.$$

where S^{m-n} is the $(m - n)$-dimensional unit sphere in $\mathbb{R}^{m-n+1}$ with standard Riemannian meetric $g_{S^{m-n}}$. By [15, 6], $Ric_{m,n}(L)$ equals to the horizontal projection of the Ricci curvature on $(M^n \times S^{m-n}, \widetilde{g})$.

Similarly to the case of Ricci flow, for any $m \in [n, \infty]$ and $K \in \mathbb{R}$, we introduce the notion of the (K, m)-Ricci flow as follows

$$\frac{\partial g}{\partial t} = -2Ric_{m,n}(L) + Kg. \tag{11}$$

The (K, m)-super Ricci flow is defined by

$$\frac{\partial g}{\partial t} \geq -2Ric_{m,n}(L) + Kg. \tag{12}$$

In the case $m = n$ and $K = 0$, (11) reduces to (9). In the case $m = \infty$, the (K, ∞)-Ricci flow reads as

$$\frac{\partial g}{\partial t} = -2(Ric + \nabla^2\phi) + Kg, \tag{13}$$

and the (K, ∞)-super Ricci flow reads as

$$\frac{\partial g}{\partial t} \geq -2(Ric + \nabla^2\phi) + Kg. \tag{14}$$

In this talk, we present some results on the differential Harnack inequality for positive solution to the heat equation associated with the Witten Laplacian

$$\partial_t u = Lu, \tag{15}$$

on complete Riemannian manifolds with $CD(K, m)$-condition and with super Ricci flows, in particular, Ricci flow and backward Ricci flow.

Theorem 0.1 *([6, 7]) Let (M, g) be a complete Riemannian manifold with C^2-potential ϕ. Suppose that for some constants $m \in [n, \infty)$ and $K \in \mathbb{R}$ we have $Ric_{m,n}(L) \geq -K$. Then for any positive solution to the heat equation* (15)*, and for any $\alpha > 1$, we have*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \leq \frac{m\alpha^2}{2t} + \frac{m\alpha^2 K}{\sqrt{2}(\alpha - 1)}. \tag{16}$$

Moreover, the parabolic Harnack inequality holds: for any $x, y \in M$ and $s < t$,

$$\frac{u(x,s)}{u(y,t)} \le \left(\frac{t}{s}\right)^{m\alpha/2} \exp\left(\frac{\alpha^2 d^2(x,y)}{4(t-s)} + \frac{m\alpha K}{2(\alpha-1)}(t-s)\right). \tag{17}$$

In particular, if $Ric_{m,n}(L) \ge 0$, we have the Li-Yau DHI

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \le \frac{m}{2t}. \tag{18}$$

Equivalently when $Ric_{m,n}(L) \ge 0$, it holds

$$L \log u + \frac{m}{2t} \ge 0. \tag{19}$$

Moreover, for any $x, y \in M$ and $s < t$, it holds

$$\frac{u(x,s)}{u(y,t)} \le \left(\frac{t}{s}\right)^{m/2} \exp\left(\frac{d^2(x,y)}{4(t-s)}\right).$$

In [6, 7], we proved the two-sides Gaussian type estimates for the fundamental solution (i.e., the heat kernel) of the Witten Laplacian, and the Varadhan small time asymptotic behavior for the heat kernel of the Witten Laplacian. As an application of (19), we introduce the H_m-entropy for the heat equation (15) as follows

$$H_m(u(t)) = -\int_M u \log u d\mu - \frac{m}{2}(\log(4\pi t) + 1)$$

and derived its monotonicity under the condition $Ric_{m,n}(L) \ge 0$, i.e.,

$$\frac{d}{dt} H_m(u(t)) = -\int_M \left(L \log u + \frac{m}{2t}\right) u d\mu \le 0.$$

Similarly to Perelman [17], using the Boltzmann entropy formula in statistical mechanics we introduced the W_m-entropy for the heat equation (15) and proved the monotonicity and rigidity theorem for W_m on complete Riemannian manifolds satisfying $Ric_{m,n}(L) \ge 0$. For details, see [7, 8].

Theorem 0.2 *([8]) Let (M, g) be a complete Riemannian manifold with C^2-potential ϕ. Suppose that for some constant $K \in \mathbb{R}$ we have $Ric(L) \ge -K$. Then, for any positive and bounded solution to the heat equation (1), we have*

$$\frac{|\nabla u|^2}{u^2} \le \frac{2K}{1 - e^{-2Kt}} \log\left(\frac{A}{u}\right), \tag{20}$$

where $A = \sup\{u(t,x) : x \in M, t \ge 0\}$.

We would like to point out that, DHI (20) is an improved version of Hamilron's DHI (6). Indeed, using the inequality

$$\frac{2K}{1 - e^{-2Kt}} \le 2K + \frac{1}{t},$$

we can derive the Hamilton DHI (6) from DHI (20).

In a joint work with Songzi Li [11], we proved the Li-Yau-Hamilton DHI for positive solution to the heat equation associated with the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition. More precisely, we have the following

Theorem 0.3 *([11]) Let (M, g) be a complete Riemannian manifold with C^2-potential ϕ. Suppose that for some constants $m \in [n, \infty)$ and $K \in \mathbb{R}$ we have $Ric_{m,n}(L) \geq -K$. Then for any positive solution to the heat equation* (15)*, we have the Li-Yau-Hamilton DHI*

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{\partial_t u}{u} \leq \frac{m}{2t}e^{4Kt}. \tag{21}$$

Moreover, the following parabolic Harnack inequality holds

$$\frac{u(x,s)}{u(y,t)} \leq \left(\frac{t}{s}\right)^{m/2} \exp\left(\frac{e^{2Ks}[1+2K(t-s)]}{4}\frac{d^2(x,y)}{t-s} + \frac{m}{2}[e^{2Kt} - e^{2Ks}]\right). \tag{22}$$

As an application of (21), we introduced the $H_{m,K}$-entropy as follows

$$H_{m,K}(u(t)) = -\int_M u \log u d\mu - \Phi_{m,K}(t),$$

where $\Phi_{m,K} \in C((0,\infty), \mathbb{R})$ is a function satisfying

$$\frac{d}{dt}\Phi_{m,K}(t) = \frac{m}{2t}e^{4Kt}.$$

Then, under the condition $Ric_{m,n}(L) \geq -K$, we have

$$\frac{d}{dt}H_{m,K}(u(t)) = \int_M \left(\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{\partial_t u}{u} - \frac{m}{2t}e^{4Kt}\right) u d\mu \leq 0.$$

Similarly to Perelman [17], using the Boltzmann entropy formula in statistical mechanics, we introduced the $W_{m,K}$-entropy for the heat equation (15) and proved its monotonicity and rigidity theorem. See [11]. See also [9, 13].

In a joint work with Songzi Li [12], see also [10, 14], we extended the DHI (20) and Hamilton's DHI (6) to positive and bounded solutions to the heat equation associated with the Witten Laplacian on complete Riemannian manifolds equipped with a $(-K, \infty)$-super Ricci flow. More precisely, we have the following

Theorem 0.4 *([10, 12, 14]) Let $(M, g(t), \phi(t), t \in [0, T])$ be a manifold equipped with a family of complete Riemannian metrics $g(t)$ and C^2-potentials $\phi(t)$, $t \in [0, T]$. Suppose that $(g(t), \phi(t), t \in [0, T])$ is a $(-K, \infty)$-super Ricci flow*

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) \geq -Kg. \tag{23}$$

Then, for any positive and bounded solution to the heat equation (15) *associated with the time dependent Witten Laplace, the DHI* (20) *holds*

$$\frac{|\nabla u|^2}{u^2} \leq \frac{2K}{1 - e^{-2Kt}} \log\left(\frac{A}{u}\right),$$

where $A = \sup\{u(t,x) : x \in M, t \geq 0\}$. Moreover, the Hamilton DHI (6) *holds, i.e.,*

$$\frac{|\nabla u|^2}{u^2} \leq \left(2K + \frac{1}{t}\right) \log(A/u).$$

In a joint work with Songzi Li [12], we also proved the Li-Yau type DHI for positive solutions to the heat equation associated with the time dependent Witten Laplacian on a variant of the backward $(-K, m)$-super Ricci flows. In particular, we proved the Li-Yau type DHI for positive solutions to the heat equation associated with the time dependent Witten Laplacian on Ricci flow and backward Ricci flow. See also [10]. To save the length of this abstract, we omit the statements of these results here.

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