ASYMPTOTIC BEHAVIOR IN TIME OF SOLUTIONS TO COMPLEX VALUED NONLINEAR KLEIN-GORDON EQUATION

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1. Abstract

In this talk, we consider the long time behavior of solutions to the initial value problem for the "complex valued" cubic nonlinear Klein-Gordon equation in one space dimension:

$$\begin{cases} (\Box+1)u=\lambda|u|^2u & t\in\mathbb{R}, x\in\mathbb{R}, \\ u(0,x)=u_0(x), \quad \partial_t u(0,x)=u_1(x) & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $\Box=\partial_t^2-\partial_x^2$ is d'Alembertian, $u:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is an unknown function, $u_0, u_1:\mathbb{R}\to\mathbb{C}$ are given functions, and λ is a non-zero real constant. The complex valued nonlinear Klein-Gordon equation/system arise in various fields of physics. For example, the nonlinear Dirac equation which is the important model in the relativistic quantum fields (see [3] for instance) can be reduced to the system of the complex valued nonlinear Klein-Gordon equations.

Since L^∞ decay rate of solution to the one dimensional linear Klein-Gordon equation is $O(|t|^{-1/2})$ as $|t|\to\infty$, from the point of view of the linear scattering theory we expect that the cubic nonlinear term is the long range type. In fact, it is well-known that the non-trivial solutions to (1.1) do not scatter to the free solution, see [4]. Therefore the asymptotic behavior in time of solution to (1.1) is different from that of the linear equation. For the real valued case, the asymptotic behavior in time of solution to (1.1) is studied by the several authors. Delort [1] obtained an asymptotic profile of a time global solution to (1.1) for the small initial data with compact support. Note that the compact support assumption in [1] is removed by Hayashi and Naumkin [5].

For the complex valued case, Sunagawa [7] proved the L^∞ decay estimate of solutions to (1.1). The main purpose of this talk is to obtain the large time asymptotics of solutions to the initial value problem (1.1). We consider the case $t\geqslant 0$ only since the case $t\leqslant 0$ can be treated in a similar way.

Our main result is as follows.

Theorem 1.1. *Let $m\geqslant 11$ be an integer. Then, there exists $\varepsilon_0>0$ with the following properties: If u_0 and u_1 are compactly supported and satisfy $\varepsilon:=\|u_0\|_{H^m}+\|u_1\|_{H^{m-1}}\leqslant\varepsilon_0$, then, there exists a unique global solution $u\in C([0,\infty);H^m(\mathbb{R}))\cap C^1([0,\infty);H^{m-1}(\mathbb{R}))$ to (1.1) which satisfies*

$$\|u(t)\|_{L_x^\infty}\leqslant C\varepsilon(1+t)^{-\frac{1}{2}} \tag{1.2}$$

for any $t \geqslant 0$. *Furthermore, there exist* $\Phi_\pm \in L^\infty(\mathbb{R})$ *such that*

$$
\begin{aligned}
u(t,x) &= \frac{1}{t^{1/2}}\Phi_+\left(\frac{x}{t}\right)\exp\left(i\sqrt{t^2-|x|^2}+i\Psi_+\left(\frac{x}{t}\right)\log t\right)\\
&\quad +\frac{1}{t^{1/2}}\Phi_-\left(\frac{x}{t}\right)\exp\left(-i\sqrt{t^2-|x|^2}+i\Psi_-\left(\frac{x}{t}\right)\log t\right)\\
&\quad +O\left(\varepsilon t^{-\frac{3}{2}+C\varepsilon}\right) \qquad \text{as } t\to\infty, \qquad (1.3)
\end{aligned}
$$

where $\Psi_\pm$ *are given by*

$$
\begin{aligned}
\Psi_+(y) &= -\frac{1}{2}\lambda\sqrt{1-|y|^2}\left(|\Phi_+(y)|^2+2\,|\Phi_-(y)|^2\right),\\
\Psi_-(y) &= \frac{1}{2}\lambda\sqrt{1-|y|^2}\left(2\,|\Phi_+(y)|^2+|\Phi_-(y)|^2\right)
\end{aligned}
$$

and C *is a positive constant independent of* ε.

From (1.3), we see that an asymptotic profile of time global solution to (1.1) is given by solution to the linear Klein-Gordon equation with a logarithmic phase correction. It is known that $\Phi_- = \overline{\Phi_+}$ for the real valued case (see [1]). Note that the authors and Uriya [6] constructed a solution to (1.1) which converge to "prescribed" final states in the sense of (1.3).

Remark 1.2. In [6], we studied large time behavior of complex-valued solutions to the Klein-Gordon equation with a gauge invariant quadratic nonlinearity in two space dimensions:

$$(\Box+1)u=\lambda|u|u \qquad t\in\mathbb{R}, x\in\mathbb{R}^2, \qquad (1.4)$$

where $\Box=\partial_t^2-\partial_{x_1}^2-\partial_{x_2}^2$ is d'Alembertian, $u:\mathbb{R}\times\mathbb{R}^2\to\mathbb{C}$ is an unknown function, and λ is a non-zero real constant. We constructed a solution to (1.4) which converges to prescribed final states, where the final state is given by the free solution with a logarithmic phase correction. Note that the logarithmic phase correction given by [6] has one more parameter which is characterized by the final data. It is an interesting open question whether all small global solutions to (1.4) behave like such an asymptotic profile.

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