Some decomposition of L^p-vector fields and its application to the Navier-Stokes equatinos

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1 Introduction.

Let Ω be a bounded domain in $\mathbb{R}^3$ with C^∞-boundary $\partial\Omega$. It is well known that every vecotor field u in $L^r(\Omega)$, $1 < r < \infty$, can be uniquely represented as

$$u = v + \nabla p, \tag{1.1}$$

where $v \in L^r(\Omega)$ with div $v = 0$ in the sense of distributions in Ω with $v \cdot \nu = 0$ on $\partial\Omega$, and $p \in W^{1,r}(\Omega)$. Here and in what follows, ν denotes the unit outer normal to $\partial\Omega$. For smooth vector fields in Ω,Weyl [27] proved such a decomposition as an orthogonal sum $L^2(\Omega)$. The case for more general L^r-vector fileds was treated by Fujiwara-Morimoto [10], Solonnikov [23] and Simader-Sohr [21]. It should be noted that (1.1) holds for all $u \in L^r(\Omega)$, so we can define the projection operator P_r by $P_r u = v$ which palys an important role for investigation into the Navier-Stokes equations. In this article, we shall prove more precise decomposition for v in (1.1):

$$v = h + \text{rot } w, \tag{1.2}$$

where $w \in W^{1,r}(\Omega)$ with $w \times \nu = 0$ on $\partial\Omega$, and where $h \in C^\infty(\bar{\Omega})$ satisfies rot $h = 0$, div $h = 0$ in Ω with $h \cdot \nu = 0$ on $\partial\Omega$. This may be regarded as the generalization of the de Rham-Hodge-Kodaira orthogonal decoposition in L^2 for C^∞ p-forms $\Lambda^p(M)$ on compact Riemaniann n-manifolds (M, g) without boundary

$$\Lambda^p(M) = H^p(M) \oplus +d(\Lambda^{p-1}(M)) \oplus \delta(\Lambda^{p+1}(M)), \quad p = 1, \cdots, n-1, \tag{1.3}$$

where d and δ denote the exterior differentation and its formal adjoint operator, respectively, and $H^p(M) = \{h \in \Lambda^p(M); dh = 0, \delta h = 0\}$. Our decoposition (1.2) holds for all $u \in L^r(\Omega)$ with $1 < r < \infty$ and for all smooth bounded domains Ω in $\mathbb{R}^3$. In the case Ω has a certain topological type, similar decomposotion to (1.2) in $L^2(\Omega)$ was investigated by Foias-Temam [8] and Yoshida-Giga [28]. However, their characterization of orthogal complement of harmonic vector fields is different from ours.

To prove (1.2), the vector potential w is formally obtained from the boudary value problem

$$\begin{cases} -\text{rot rot } w = \text{rot } u & \text{in } \Omega, \\ w \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

It should be noted that this is not an elliptic system for w. Hence, to recover ellipticity, we need to impose on w the following additinal condition:

$$(1.4) \qquad \begin{cases} -\text{rot rot } w = \text{rot } u & \text{in } \Omega, \\ \text{div } w = 0 & \text{in } \Omega, \\ w \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Unfortunately, this modified system is not an ellipic boundary value problem in the sense of Agmon-Douglis-Nirenberg [1]. Indeed, if $w \in W^{2,r}(\Omega)$ for some $1 < r < \infty$, then we may rewrite (1.4) as

$$(1.5) \qquad \begin{cases} -\Delta w = \text{rot } u & \text{in } \Omega, \\ \text{div } w = 0 & \text{on } \partial\Omega, \\ w \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

which can be treated as an ellipic boudary value problem in the sense of Agmon-Douglis-Nirenberg. Since we need to solve (1.4) for an arbitrary given $u \in L^r(\Omega)$, we can expect only that $w \in W^{1,r}(\Omega)$, so the value div w on the boudary $\partial\Omega$ in (1.5) cannot be always well-defined. This means that we are not able to apply to (1.4) the fully established theory on existence and reglarity of solutions to the elliptic boundary value problems. To get around such difficulty, we shall formulate (1.4) in a weak sense such as to find $w \in W^{1,r}(\Omega)$ satisfying

$$(1.6) \qquad \int_\Omega \text{rot } w \cdot \text{rot } \Phi dx = \int_\Omega u \cdot \text{rot } \Phi dx$$

for all $\Phi \in W^{1,r'}(\Omega)$ with div $\Phi = 0$ in Ω, $\Phi \times \nu = 0$ on $\partial\Omega$, where $r' = r/(r-1)$. This procedure is similar to that of finding a scalar potential p in (1.1). Indeed, $p \in W^{1,r}(\Omega)$ is obtained from the weak solution of the Neumann boundary problem for Δ in Ω.

$$(1.7) \qquad \int_\Omega \nabla p \cdot \nabla \phi dx = \int_\Omega u \cdot \nabla \phi dx \quad \text{for all } \phi \in W^{1,r'}(\Omega).$$

Simader-Sohr [21] solved (1.7) by introducing a variational inequality in $W^{1,r}(\Omega)$ which is a variant of coercive estimate of Dirichelt form accociated to the operator $-\Delta$. Our proof for solvability of (1.4) is also based on the following variational inequality. In fact, we shall show that for every $1 < r < \infty$, there is a constant C such that

$$
\begin{aligned}
&\|w\|_{W^{1,r}(\Omega)} \\
(1.8)\quad &\le C \quad \sup\left\{\frac{\left|\int_\Omega \text{rot } w\cdot \text{rot } \Phi dx\right|}{\|\Phi\|_{W^{1,r'}(\Omega)}}; \Phi\in W^{1,r'}(\Omega), \text{div } \Phi = 0 \text{ in } \Omega, \ \Phi\times\nu = 0 \text{ on } \partial\Omega\right\} \\
&\quad + \sum_{i=1}^{L}\left|\int_\Omega w\cdot\psi_i dx\right|
\end{aligned}
$$

holds for all $w\in W^{1,r}(\Omega)$ with div $w=0$ in Ω, $w\times\nu=0$ on $\partial\Omega$, where $\{\psi_1,\cdots,\psi_L\}$ is a basis of the finite dimensional space $V_{har}(\Omega)=\{\psi\in C^\infty(\bar{\Omega}); \text{rot } \psi = 0, \text{div } \psi = 0 \text{ in } \Omega, \psi\times\nu|_{\partial\Omega}=0\}$. If $\partial\Omega$ consists of $L+1$ connected components $\Gamma_0,\Gamma_1,\cdots,\Gamma_L$ of disjonit surfaces with $\Gamma_1,\cdots,\Gamma_L$ inside of Γ_0, i.e., $\partial\Omega=\cup_{0=1}^{L}\Gamma_i$, then we have dim.$V_{har}(\Omega)=L$. Similar investigation into the variational inequality was done by Greisinger [13] in the case when Ω is a star-shaped domain. Compared with our situtation, she treated the special case when $V_{har}(\Omega)=\{0\}$. Furthermore, since she took w in $W^{1,r}(\Omega)$ with $w=0$ on $\partial\Omega$, it seems to be an open question whether the complement of the space (1.2) coinsides with the vector space ∇p with the scalar potential $p\in W^{1,r}(\Omega)$ as in (1.1).

As an application of (1.2), we shall show the generalized Biot-Savard law for the vector field u in $W^{1,r}(\Omega)$ with $u\cdot\nu=0$ on $\partial\Omega$. In the whole space $\mathbb{R}^3$, if $v\in W^{1,r}(\mathbb{R}^3)$ with div $v=0$, then v can be represented as

$$
v(x)=\int_{\mathbb{R}^3} K(x-y)\times \text{rot } v(y)dy, \quad K(x) = -\frac{1}{4\pi}\frac{x}{|x|^3}
$$

for all $x\in\mathbb{R}^3$. Since $\nabla K(x)$ is a Calderon-Zygmund kernel, there holds

$$
\begin{aligned}
&\|\nabla v\|_{L^r(\mathbb{R}^3)} \le C\|\text{rot } v\|_{L^r(\mathbb{R}^3)}, \\
&\|\nabla v\|_{L^\infty(\mathbb{R}^3)} \le C\{1+\|\text{rot } v\|_{BMO}\log(e+\|v\|_{H^3(\mathbb{R}^3)})\}.
\end{aligned}
$$

See e.g., Beale-Kato-Majda [2] and Kozono-Taniuchi [15]. In bounded domains Ω in $\mathbb{R}^3$, our decomposition (1.2) gives the corresponding estimates

$$
\begin{aligned}
(1.9)\qquad & \|\nabla u\|_{L^r(\Omega)} \le C(\|\text{div } u\|_{L^r(\Omega)} + \|\text{rot } u\|_{L^r(\Omega)} + \|u\|_{L^1(\Omega)}), \\
(1.10)\qquad & \|\nabla u\|_{L^\infty(\Omega)} \\
\le\ & C\left\{1+\|u\|_{L^r(\Omega)} + (\|\text{div } u\|_{bmo}+\|\text{rot } u\|_{bmo})\log(e+\|u\|_{W^{s,r}(\Omega)})\right\}, \quad s>1+3/r
\end{aligned}
$$

provied u satisfies $u\cdot\nu=0$ on $\partial\Omega$. By means of the reprentation formula for $u\in W^{1,r}(\Omega)$ given by Kress [17], von Wahl [26] obtained (1.9) without $\|u\|_{L^1(\Omega)}$ on the right hand side when h always vanishes in (1.2). On the other hand, if $u\in W^{1,r}(\Omega)$ with $u\cdot\nu=0$ on $\partial\Omega$, then our decomposition (1.2) yields necessarily such a reprsentation formula as we can deduce (1.9) immediately. Since we need not impose any assumption on Ω, our estimate (1.9) may be regared as generalization of von Wahl [26]. Furthremore, we shall show the corresponding estimate for u in higher order Sobolev space $W^{s,r}(\Omega)$ via div u and rot u in $W^{s-1,r}(\Omega)$ even though $u\cdot\nu$ or

$u \times \nu$ does not vanish on $\partial\Omega$. Concerning (1.10), by introducing a different elliptic system from (1.5), Ferrari [7], Shirota-Yanagisawa [22] and Ogawa-Taniuchi [19] gave the proof for u with $\mathrm{div}\ u = 0$, $u \cdot \nu|_{\partial\Omega} = 0$. On the other hand, we shall see that these estimates can be derived directly from regularity theorem of weak solutions to (1.5). Since we need to impose neither assumption on Ω nor $\mathrm{div}\ u = 0$, our estimates (1.9) and (1.10) may be regared as generalization of [26], [7], [22] and [19].

Based on (1.10), they [7], [22], [19] proved an extension criterion via vorticity $\mathrm{rot}\ v$ on local strong solutions v for the incompressible Euler equations. In article, we shall show the correponding criterion for the Navier-Stokes equations. Compared with the Euler equations, we need to give a bound of boundary integral on $\partial\Omega$ for $-\Delta v$ to the solution v of the Navier-Stokes equations. To avoid such difficulty, we shall make use of (1.10) together with the estimate of fractional powers of the Stokes operator in $L^r(\Omega)$. Indeed, we shall show that if the strong solution v of the Navier-Stokes equatons on $\Omega \times (0,T)$ satisfies

$$\int_0^T \|\mathrm{rot}\ v(t)\|_{bmo} dt < \infty, \tag{1.11}$$

then v can be extended to the solution on $\Omega \times (0, T')$ for some $T' > T$.

2 Results.

Let us first impose the following assumption on the domain Ω:

Assumption. Ω is a bounded domain in $\mathbb{R}^3$ with the C^∞-boundary $\partial\Omega$.

Before stating our results, we introduce some function spaces. Let $C^\infty_{0,\sigma}(\Omega)$ denote the set of all C^∞-vector functions $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ with compact support in Ω, such that $\mathrm{div}\ \varphi = 0$. $L^r_\sigma(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ denotes the duality pairing between $L^r(\Omega)$ and $L^{r'}(\Omega)$, where $1/r + 1/r' = 1$. $L^r(\Omega)$ stands for the usual (vector-valued) L^r-space over Ω, $1 < r < \infty$. Let us recall the generalized trace theorem for $u \cdot \nu$ and $u \times \nu$ on $\partial\Omega$ defined on the spaces $E^r_{div}(\Omega)$ and $E^r_{rot}(\Omega)$, respectively.

$$\begin{aligned} E^r_{div}(\Omega) &\equiv \{u \in L^r(\Omega); \mathrm{div}\ u \in L^r(\Omega)\} \text{ with the norm } \|u\|_{E^r_{div}} = \|u\|_r + \|\mathrm{div}\ u\|_r\}, \\ E^r_{rot}(\Omega) &\equiv \{u \in L^r(\Omega); \mathrm{rot}\ u \in L^r(\Omega)\} \text{ with the norm } \|u\|_{E^r_{rot}} = \|u\|_r + \|\mathrm{rot}\ u\|_r\}. \end{aligned}$$

It is known that there are bounded operators γ_ν and τ_ν on $E^r_{div}(\Omega)$ and $E^r_{rot}(\Omega)$ with properties that

$$\begin{aligned} \gamma_\nu : u \in E^r_{div}(\Omega) \mapsto \gamma_\nu u \in W^{1-1/r',r'}(\partial\Omega)^*, \qquad \gamma_\nu u = u \cdot \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}), \\ \tau_\nu : u \in E^r_{rot}(\Omega) \mapsto \tau_\nu u \in W^{1-1/r',r'}(\partial\Omega)^*, \qquad \tau_\nu u = u \times \nu|_{\partial\Omega} \text{ if } u \in C^1(\bar{\Omega}), \end{aligned}$$

respectively. We have the following the generalized Stokes formula

$$(u, \nabla p) + (\mathrm{div}\ u, p) = \langle \gamma_\nu u, \gamma_0 p \rangle_{\partial\Omega} \qquad \text{for all } u \in E^r_{div}(\Omega) \text{ and all } p \in W^{1,r'}(\Omega), \tag{2.1}$$

$$(u, \mathrm{rot}\ \phi) = (\mathrm{rot}\ u, \phi) + \langle \tau_\nu u, \gamma_0 \phi \rangle_{\partial\Omega} \qquad \text{for all } u \in E^r_{rot}(\Omega) \text{ and all } \phi \in W^{1,r'}(\Omega), \tag{2.2}$$

where γ_0 denotes the usual trace operator from $W^{1,r'}(\Omega)$ onto $W^{1-1/r',r'}(\partial\Omega)$, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ is the duality paring between $W^{1-1/r',r'}(\partial\Omega)^*$ and $W^{1-1/r',r'}(\partial\Omega)$. Notice that $L^r_\sigma(\Omega) = \{u \in L^r(\Omega); \mathrm{div}\ u = 0 \text{ in } \Omega \text{ with } \gamma_\nu u = 0\}$.

Let us define two spaces $X^r(\Omega)$ and $V^r(\Omega)$ for $1 < r < \infty$ by

$$X^r(\Omega) \equiv \{u \in L^r(\Omega); \mathrm{div}\ u \in L^r(\Omega), \mathrm{rot}\ u \in L^r(\Omega), \gamma_\nu u = 0\}, \tag{2.3}$$

$$V^r(\Omega) \equiv \{u \in L^r(\Omega); \mathrm{div}\ u \in L^r(\Omega), \mathrm{rot}\ u \in L^r(\Omega), \tau_\nu u = 0\}. \tag{2.4}$$

Equipped with the norms $\|u\|_{X^r}$ and $\|u\|_{V^r}$

$$\|u\|_{X^r}, \|u\|_{V^r} \equiv \|\mathrm{div}\ u\|_r + \|\mathrm{rot}\ u\|_r + \|u\|_r, \tag{2.5}$$

we may regard $X^r(\Omega)$ and $V^r(\Omega)$ as Banach spaces. Indeed, in Thereom 2 below, we shall see that both $X^r(\Omega)$ and $V^r(\Omega)$ are closed subspaces in $W^{1,r}(\Omega)$ since it holds

$$\|\nabla u\|_r \le C\|u\|_{X^r} \quad \text{for all } u \in X^r(\Omega) \quad \text{and} \quad \|\nabla u\|_r \le C\|u\|_{V^r} \quad \text{for all } u \in V^r(\Omega), \tag{2.6}$$

respectively, where $C = C(r)$ is a constant depending only on r. Furthermore, we define $X^r_\sigma(\Omega)$ and $V^r_\sigma(\Omega)$ by

$$X^r_\sigma(\Omega) \equiv \{u \in X^r(\Omega); \mathrm{div}\ u = 0 \quad \text{in } \Omega\}, \quad V^r_\sigma(\Omega) \equiv \{u \in V^r(\Omega); \mathrm{div}\ u = 0 \quad \text{in } \Omega\}. \tag{2.7}$$

Finally, we denote by $X^r_{har}(\Omega)$ and $V^r_{har}(\Omega)$ the L^r-spaces of harmonic vector fileds on Ω

$$X^r_{har}(\Omega) \equiv \{u \in X^r_\sigma(\Omega); \mathrm{rot}\ u = 0\}, \quad V^r_{har}(\Omega) \equiv \{u \in V^r_\sigma(\Omega); \mathrm{rot}\ u = 0\}. \tag{2.8}$$

Our main result now reads

Theorem 1 *Let Ω be as in the Assumption. Suppose that $1 < r < \infty$.*

(1) It holds

$$\begin{aligned} X^r_{har}(\Omega) &= \{h \in C^\infty(\bar\Omega); \mathrm{div}\ h = 0, \mathrm{rot}\ h = 0 \textit{ in } \Omega \textit{ with } h\cdot\nu = 0 \textit{ on } \partial\Omega\}(\equiv X_{har}(\Omega)), \\ V^r_{har}(\Omega) &= \{h \in C^\infty(\bar\Omega); \mathrm{div}\ h = 0, \mathrm{rot}\ h = 0 \textit{ in } \Omega \textit{ with } h\times\nu = 0 \textit{ on } \partial\Omega\}(\equiv V_{har}(\Omega)). \end{aligned}$$

Both $X_{har}(\Omega)$ and $V_{har}(\Omega)$ are of finite dimensional vector space.

(2) For every $u \in L^r(\Omega)$, there are $p \in W^{1,r}(\Omega)$, $w \in V^r_\sigma(\Omega)$ and $h \in X_{har}(\Omega)$ such that u can be represented as

$$u = h + \mathrm{rot}\ w + \nabla p. \tag{2.9}$$

Such a triplet $\{p, w, h\}$ is subordinate to the estimate

$$\|\nabla p\|_r + \|w\|_{V^r} + \|h\|_r \le C\|u\|_r \tag{2.10}$$

with the constant $C = C(r)$ independent of u. The above decompostion (2.9) is unique. In fact, if u has another expression

$$u = \tilde h + \mathrm{rot}\ \tilde w + \nabla\tilde p$$

for $\tilde h \in X_{har}(\Omega)$, $\tilde w \in V^r_\sigma(\Omega)$ and $\tilde p \in W^{1,r}(\Omega)$, then we have

$$h = \tilde h, \quad \mathrm{rot}\ w = \mathrm{rot}\ \tilde w, \quad \nabla p = \nabla\tilde p. \tag{2.11}$$

An immediate consequence of the above theorem is

Corollary 1 *Let Ω be as in the Assumption. By the unique decomposition (2.9) we have*

$$L^r(\Omega) = X_{har}(\Omega) \oplus \text{rot } V^r_\sigma(\Omega) \oplus \nabla\ W^{1,r}(\Omega), \quad 1 < r < \infty. \quad \textit{(direct sum)} \tag{2.12}$$

Let S_r, R_r and Q_r be projection operators associated to (2.9) from $L^r(\Omega)$ onto $X_{har}(\Omega)$, rot $V^r_\sigma(\Omega)$ and $\nabla\ W^{1,r}(\Omega)$, respectively, i.e.,

$$S_r u \equiv h, \quad R_r u \equiv \text{rot } w, \quad Q_r u \equiv \nabla p. \tag{2.13}$$

Then we have

$$\|S_r u\|_r \le C\|u\|_r, \quad \|R_r u\|_r \le C\|u\|_r, \quad \|Q_r u\|_r \le C\|u\|_r \tag{2.14}$$

for all $u \in L^r(\Omega)$, where $C = C(r)$ is the constant depending only on $1 < r < \infty$. Moreover, there holds

$$\begin{cases} S_r^2 = S_r, & S_r^* = S_{r'}, \\ R_r^2 = R_r, & R_r^* = R_{r'} \\ Q_r^2 = Q_r, & Q_r^* = Q_{r'}, \end{cases} \tag{2.15}$$

where S_r^*, R_r^* and Q_r^* denote the adjoint operators on $L^{r'}(\Omega)$ of S_r, R_r and Q_r, respectively.

Remark 1. (1) It is known that

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus \nabla\ W^{1,r}(\Omega), \quad 1 < r < \infty, \quad \text{(direct sum)}. \tag{2.16}$$

See Fujiwara-Morimoto [10], Solonnikov [23] and Simader-Sohr [21]. Our decomposition (2.12) gives a more precise direct sum of $L^r_\sigma(\Omega)$ such as

$$L^r_\sigma(\Omega) = X_{har}(\Omega) \oplus \text{rot } V^r_\sigma(\Omega), \quad 1 < r < \infty. \quad \text{(direct sum)} \tag{2.17}$$

(2) Suppose that the boundary $\partial\Omega$ has $L + 1$ connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$ of C^2-surfaces such that $\Gamma_1, \cdots, \Gamma_L$ lie inside of Γ_0 with $\Gamma_i \cap \Gamma_j = \phi$ for $i \ne j$, and scuh that

$$\partial\Omega = \bigcup_{j=0}^{L} \Gamma_j. \tag{2.18}$$

Moreover, we assume that there are N C^2-surfaces $\Sigma_1, \cdots, \Sigma_N$ such that $\Sigma_i \cap \Sigma_j = \phi$ for $i \ne j$, and such that

$$\dot{\Omega} \equiv \Omega \setminus \Sigma, \Sigma \equiv \bigcup_{j=1}^{N} \Sigma_j \quad \text{is simply connected.} \tag{2.19}$$

Then Foias-Temam [8] showed that

$$\text{dim.} X_{har}(\Omega) = N. \tag{2.20}$$

They [8] also gave an orthogonal decompostion of $L^2_\sigma(\Omega)$ such as

$$L^2_\sigma(\Omega) = X_{har}(\Omega) \oplus H_1(\Omega) \quad \text{(orthogonal sum in } L^2(\Omega)),$$

where

$$H_1(\Omega) \equiv \{u \in L^2_\sigma(\Omega); \int_{\Sigma_j} u \cdot \nu dS = 0, \quad j = 1, \cdots, N\}.$$

Yoshida-Giga [28] investigated the operator rot with its domain $D(\mathrm{rot}) = \{u \in H_1(\Omega); \mathrm{rot}\ u \in H_1(\Omega)\}$ and showed that $H_1(\Omega) = \mathrm{rot}\ V^2_\sigma(\Omega)$. Furthermore, they [28] gave another type of orthognal L^2-decomposition of vector fileds $u \in D(\mathrm{rot})$. From our decomposition (2.17) with $r = 2$, it follows also that $H_1(\Omega) = \mathrm{rot}\ V^2_\sigma(\Omega)$.

(3) In the case when Ω is a star-shaped domain, Griesinger [13] gave a simlar decomposition in $L^r(\Omega)$ for $1 < r < \infty$. In her case, it holds $N = 0$. Since she took the smaller space $W^{1,r}_0(\Omega)$ than our space $V^r(\Omega)$, it seems to be an open question whether, in the same way as in (2.12), the anihilator rot $W^{1,r}_0(\Omega)^\perp$ of rot $W^{1,r}_0(\Omega)$ in $L^{r'}(\Omega)$ coinsides with $\nabla\ W^{1,r'}(\Omega)$.

As an application of our decomposition, we have the following gradeint and higher order estimates of vector fields via div and rot .

Theorem 2 *Let Ω be as in the Assumption. Suppose that $1 < r < \infty$.*
(1) *(in case γ_ν) Let dim.$X_{har}(\Omega) = N$ and let $\{\phi_1, \cdots \phi_N\}$ be a basis of $X_{har}(\Omega)$.*
(i) *It holds $X^r(\Omega) \subset W^{1,r}(\Omega)$ with the estimate*

$$\|\nabla u\|_r + \|u\|_r \le C(\|\mathrm{div}\ u\|_r + \|\mathrm{rot}\ u\|_r + \sum_{j=1}^N |(u, \phi_j)|) \quad \text{for all } u \in X^r(\Omega), \tag{2.21}$$

where $C = C(\Omega, r)$.
(ii) *Let $s \ge 1$. Suppose that $u \in L^r(\Omega)$ with* div $u \in W^{s-1,r}(\Omega)$, rot $u \in W^{s-1,r}(\Omega)$ *and $\gamma_\nu u \in W^{s-1/r,r}(\partial\Omega)$. Then we have $u \in W^{s,r}(\Omega)$ with the estimate*

$$\begin{aligned} &\|u\|_{W^{s,r}(\Omega)} \\ &\le C(\|\mathrm{div}\ u\|_{W^{s-1,r}(\Omega)} + \|\mathrm{rot}\ u\|_{W^{s-1,r}(\Omega)} + \|\gamma_\nu u\|_{W^{s-1/r,r}(\partial\Omega)} + \sum_{j=1}^N |(u, \phi_j)|), \end{aligned} \tag{2.22}$$

where $C = C(\Omega, r)$.
(2) *(in case $\tau_\nu u$) Let dim.$V_{har}(\Omega) = L$ and let $\{\psi_1, \cdots \psi_L\}$ be a basis of $V_{har}(\Omega)$.*
(i) *It holds $V^r(\Omega) \subset W^{1,r}(\Omega)$ with the estimate*

$$\|\nabla u\|_r + \|u\|_r \le C(\|\mathrm{div}\ u\|_r + \|\mathrm{rot}\ u\|_r + \sum_{j=1}^L |(u, \psi_j)|) \quad \text{for all } u \in V^r(\Omega), \tag{2.23}$$

where $C = C(\Omega, r)$.
(ii) *Let $s \ge 1$. Suppose that $u \in L^r(\Omega)$ with* div $u \in W^{s-1,r}(\Omega)$, rot $u \in W^{s-1,r}(\Omega)$ *and $\tau_\nu u \in W^{s-1/r,r}(\partial\Omega)$. Then we have $u \in W^{s,r}(\Omega)$ with the estimate*

$$\begin{aligned} &\|u\|_{W^{s,r}(\Omega)} \\ &\le C(\|\mathrm{div}\ u\|_{W^{s-1,r}(\Omega)} + \|\mathrm{rot}\ u\|_{W^{s-1,r}(\Omega)} + \|\tau_\nu u\|_{W^{s-1/r,r}(\partial\Omega)} + \sum_{j=1}^L |(u, \psi_j)|), \end{aligned} \tag{2.24}$$

where $C = C(\Omega, r)$.
(3) (L^∞-gradient bould) Let $u \in W^{s,r}(\Omega)$ for $s > 1 + 3/r$ with $u \cdot \nu|_{\partial\Omega} = 0$ or $u \times \nu|_{\partial\Omega} = 0$. Then we have $\nabla u \in L^\infty$ with the estimate

$$\|\nabla u\|_\infty \le C\left\{1 + \|u\|_r + (\|\text{div } u\|_{bmo} + \|\text{rot } u\|_{bmo})\log(e + \|u\|_{W^{s,r}(\Omega)})\right\}, \tag{2.25}$$

where $C = C(r)$ is the constant independent of u. For definition of the bmo-norm, see Remark 2 below.

Remark 2. (1) Let us recall the *bmo*-norm in Ω. For $f \in L^1_{loc}(\mathbb{R}^3)$, we define $\|f\|_{bmo(\mathbb{R}^3)}$ by

$$\|f\|_{bmo(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3, 0<R<1} \frac{1}{|B_R(x)|}\int_{B_R(x)} |f(y) - f_{B_R(x)}|dy + \sup_{x\in\mathbb{R}^3} \frac{1}{|B_1(x)|}\int_{B_1(x)} |f(y)|dy$$

with $f_{B_R(x)} = \dfrac{1}{|B_R(x)|}\displaystyle\int_{B_R(x)} f(y)dy$, where $B_R(x)$ denotes the ball in $\mathbb{R}^3$ centered at x with radius R and $|B_R(x)|$ is its volume. For $g \in L^1_{loc}(\Omega)$ we say $g \in bmo(\Omega)$ if there is an extension $f \in bmo(\mathbb{R}^3)$ such that $g = f$ on Ω. The *bmo*-norm $\|g\|_{bmo}$ of g on Ω is defined by

$$\|g\|_{bmo} \equiv \inf\{|f\|_{bmo(\mathbb{R}^3)}; f \in bmo(\mathbb{R}^3), f = g \quad \text{on } \Omega\}.$$

(2) von Wahl [26] treated the homogeneous gradeint bound such as

$$\|\nabla u\|_r \le C(\|\text{div } u\|_r + \|\text{rot } u\|_r)$$

for $u \in W^{1,r}(\Omega)$ with $\gamma_\nu u = 0$ and $\tau_\nu u = 0$. He proved that such a homogeneous estimate holds if and only if $N = 0$, i.e., Ω is simply connected in the case $\gamma_\nu u = 0$, and if and only if $L = 0$, i.e., Ω has only one connected component of the boundary $\partial\Omega$ in the case $\tau_\nu u = 0$, respectively. Our variational inequality (1.8) makes it possible to prove (2.21) and (2.23) for an arbitraly bounded domain Ω. So, von Wahl's estimate [26] may be regarded as a special case of ours since our Assumption on Ω includes such cases as (2.18) and (2.19) . His method is based on the representation formula for $u \in W^{1,r}(\Omega)$ via div u and rot u which is different from ours. Similar estimate to (2.22) with $\sum_{j=1}^N |(u, \phi_j)|$ replaced by $\|u\|_r$ was obtained by Temam[25, Proposition 1.4, Appendix I] for $s \ge 1$, $r = 2$ and by Bourguignon-Brezis [4, Lemma 5] for $s \ge 2$, $1 < r < \infty$, respectively. See also Duvaut-Lions [6, Theorem 6.1, Chapther 7].

(3) In $\mathbb{R}^3$, by means of the Biot-Savard law, Beale-Kato-Majda [2] and Kozono-Taniuchi [15] obtained a similar estimate to (2.25) for $u \in W^{s,r}(\mathbb{R}^3)$ with $s > 1 + 3/r$. More generalized version in the homogeneou Besov space $\dot{B}^0_{\infty,\infty}$ is found in Kozono-Ogawa-Taniuch [16]. In the case of simply connected bounded domains Ω in $\mathbb{R}^3$, Ferrari showed (2.25) for div $u = 0$ with $u \cdot \nu|_{\partial\Omega} = 0$. More general case such as (2.18) and (2.19) was treated by Shirota-Yanagisawa [22] and Ogawa-Taniuchi [19].

As an application of Theorem 2, we show an extension criterion of strong solutions of the nonstationary Navier-Stokes equations.

$$\text{(N-S)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } x \in \Omega,\ t > 0, \\ \text{div } u = 0 & \text{in } x \in \Omega,\ t > 0, \\ u = 0 & \text{on } x \in \partial\Omega,\ t > 0, \\ u|_{t=0} = a \end{cases}$$

It is shown by Fujita-Kato [9], Giga-Miyakawa [11] and Kato[14] that for every $a \in L^r_\sigma$ with $3 \le r < \infty$, there are $T > 0$ and a unique solution u of (N-S) on $(0,T)$ such that

$$u \in C([0,T); L^r_\sigma(\Omega)) \cap C^1((0,T); L^r_\sigma(\Omega)) \cap C((0,T); W^{2,r}(\Omega)). \tag{2.26}$$

It is an interesting question whether the solutions $u(t)$ blows up at $t = T$ or can be continued beyound T. Our second result now reads

Theorem 3 *Let Ω be as in the Assumption. Suppose that u is the solution of (N-S) on $(0,T)$ in the class (2.26). If*

$$\int_0^T \|\mathrm{rot}\ u(t)\|_{bmo} dt < \infty, \tag{2.27}$$

then there is $T' > T$ such that u can be contiuned to the solution of (N-S) on $(0,T')$ in the same class as (2.27).

An immediate consequense of the above theorem is

Corollary 2 *Let Ω be as in the Assumption. Suppose that u is the solution of (N-S) on $(0,T)$ in the class (2.26). If T is the maximal, then we have*

$$\lim_{t \to T-0} \int_0^t \|\mathrm{rot}\ u(s)\|_{bmo} ds = \infty. \tag{2.28}$$

In particular, it holds $\limsup_{t \to T-0} \|\mathrm{rot}\ u(t)\|_{bmo} = \infty$.

Remark 3. (1) For the Euler equations in the whole space $\mathbb{R}^3$, Beale-Kato-Majda [2] first proved the above extension criteron in the case $\int_0^T \|\mathrm{rot}\ u(t)\|_{L^\infty} dt < \infty$. Later on, Kozono-Tanuich [15] and Kozono-Taniuchi-Ogawa [16] proved it under the weaker norm such as BMO and the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$. It should be noted that there hlods the continuous inclusion $L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}$.

(2) For the Euler equations in bouded domains Ω in $\mathbb{R}^3$, Ferrari [7] treated the case when Ω is simply connected and obtained the same criterion as Beale-Kato-Majda [2] in $\mathbb{R}^3$. Shirota-Yanagisawa [22] succeeded to deal with the genaral case of multi-connected domains Ω such as (2.18) and (2.19). Then Ogawa-Taniuch [19] improved their results in terms of weaker norm bmo on Ω.

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