REMARK ON UNIQUENESS OF WEAK SOLUTION WITH ENERGY INEQUALITY FOR THE SOBOLEV CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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We consider the uniqueness of solution in $L^\infty(\mathbb{R}; H^1)$ for the Cauchy problem of the Sobolev critical nonlinear Schrödinger equation on $\mathbb{R}^n$, $n \geq 3$.

$$i\partial_t u + \Delta u = |u|^{4/(n-2)}u, \qquad (t,x) \in \mathbf{R}^{1+n}, \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbf{R}^n. \tag{2}$$

The existence of solution $L^\infty(\mathbb{R}; H^1)$ for (1)-(2) can easily be showed by combining the standard compactness argument and the energy inequality (see, e.g., [1]). Recently, the existence in $C(\mathbb{R}; H^1)$ has been proved by Colliander, Keel, Staffilani, Takaoka and Tao [2] for $n = 3$. Furthermore, the uniqueness of solution in $C(\mathbb{R}; H^1)$ is already known for $n \geq 3$ (see, e.g., [1], [3] and [5]). However, it still remains open whether the solution in $L^\infty(\mathbb{R}; H^1)$ is unique or not. In [6], Struwe gives a partial answer to this problem. Namely, he proves that the solution u in $L^\infty(\mathbb{R}; H^1) \cap L^1_{loc}(\mathbb{R}; H^2)$ is unique among solutions in $L^\infty(\mathbb{R}; H^1)$ satisfying the strong energy inequality:

$$E(u(t)) \leq E(u(s)) \qquad (t > s \geq 0 \text{ or } t < s \leq 0), \tag{3}$$

where

$$E(u) = \|\nabla u\|_{L^2}^2 + \frac{n-2}{n}\|u\|_{L^{2n/(n-2)}}^{2n/(n-2)}.$$

We note that the solution u in $L^\infty(\mathbb{R}; H^1)$ always satisfies the L^2 norm conservation law:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \qquad t \in \mathbb{R}.$$

An interesting question is what additional conditions ensure the uniqueness of solution in $L^\infty(\mathbb{R}; H^1)$. This is one of the most fundamental problems which appear in various nonlinear partial differential equations. For example, this kind of problem has a long history for the incompressible Navier-Stokes equations (see, e.g., [4] and references therein).

We have the following theorem.

Theorem 1. *Let* $n \geq 3$ *and* $u_0 \in H^1$. *Suppose that for some* $T_1, T_2 > 0$, *(1)-(2) has a solution* v *in* $C((-T_1, T_2); H^1)$. *Let* u *be a solution of (1)-(2) in* $L^\infty((-T_1, T_2); H^1)$ *with the same initial data as* v *above, which satisfies the energy inequality.*

$$(4) \qquad E(u(t)) \leq E(u_0), \qquad t \in (-T_1, T_2).$$

Then, $u(t) = v(t)$ *on* $(-T_1, T_2)$.

Remark 1. (i) Inequality (4) is often called the weak energy inequality, because (4) is weaker than (3). The standard compactness argument always yields this weak energy inequality (4), while the proof of (3) requires some more arguments.

(ii) The solution u in $L^\infty((-T_1, T_2); H^1)$ of (1)-(2) is necessarily a weakly continuous H^1-valued function.

Since the existence of solution in $C(\mathbb{R}; H^1)$ to (1)-(2) is proved for $n = 3$ in [2], the following corollary follows from the combination of Theorem 1 above and the uniqueness theorem of strongly continuous solution in H^1 by [1] and [5].

Corollary 2. *Assume that* $n = 3$ *and* $u_0 \in H^1$. *Then, a solution in* $L^\infty(\mathbb{R}; H^1)$ *of (1)-(2) satisfying (4) is unique.*

Remark 2. For the proof of Theorem 1, we show the strong continuity in time of solution in $L^\infty((-T_1, T_2); H^1)$ satisfying (4), which implies the desired uniqueness result. Theorem 1 itself seems interesting, though there may be nothing novel in its proof.

References

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