On local and global well-posedness of the Schrödinger-improved Boussinesq system

Takafumi Akahori

Department of Mathematics, Kyoto University,
Kyoto 606-8502, Japan
E-mail: akahori@math.kyoto-u.ac.jp

In the theory of Langmuir plasma turbulence, the Schrödinger equation with a self-consistent potential

$$iu_t + \Delta u - Vu = 0 \tag{1}$$

appears (cf. [5] and references therein), where $u\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ and the self-consistent potential V obeys a non-uniform wave equation

$$(\partial_t^2 - \Delta)V = \Delta|u|^2, \tag{2}$$

where $V\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. The system (1)–(2) is known as the Zakharov system. The system is hard to treat, in particular, in multi-dimension. Thus we consider a substitute for the equation (2) :

$$(\partial_t^2 - \Delta)V - \Delta\partial_t^2 V = \Delta|u|^2, \tag{3}$$

where $V\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. The equation (3) is obtained in a way using the dispersion formula

$$\omega^2 = \frac{k^2}{1+k^2}$$

which approaches the dispersion formula for the wave equation $\omega^2 = k^2$ when $k \ll 1$. The system (1)–(3) is called the Schrödinger-improved Boussinesq system.

Our aim is to prove local and global well-posedness of the Cauchy problem for (1)–(3) with data

$$(u, V, \partial_t V)|_{t=0} = (u_0, V_0, V_1) \in H^{s_1} \times H^{s_2} \times H^{s_2} \cap \dot{H}^{-1}, \tag{4}$$

and compare the result with that for the Zakharov system (1)–(2).

It is convenient to introduce the potential ϕ by the formula $\phi = i\omega(D_x)^{-1}\partial_t V + V$, where $\omega(D_x)$ is the Fourier multiplier operator with the symbol $\omega(k)^2 = \frac{k^2}{1+k^2}$. We see that ϕ obeys the equation

$$\begin{cases} i\partial_t\phi - \omega(D_x)\phi &= \omega(D_x)|u|^2, \\ V &= \mathrm{Re}\,[\phi]. \end{cases} \tag{5}$$

Then the initial condition (4) becomes

$$(u, \phi)|_{t=0} = (u_0, \phi_0) \in H^{s_1} \times H^{s_2} \tag{6}$$

with $\phi_0 = i\omega(D_x)^{-1}V_1 + V_0$.

•Main and known results

Our main results are the following two theorems.

Theorem 1 (Global well-posedness in L^2)
The Cauchy problem (1)–(5) with data (6) is globally well-posed, if $s_1 = s_2 = 0$ *and* $n \leq 2$.

In [6], Ozawa and Tsutaya proved the global well-posedness in the energy space ($s_1 = 1$, $s_2 = 0$) when $n \leq 2$. We remark that, in the 2 dimensional Zakharov system, a blow-up solution exists (cf [3]).

Theorem 2 (Local well-posedness below L^2)
The Cauchy problem (1)–(5) with data (6) is locally well-posed, if $-1/4 < s_1 \leq 0$, $-1/2 < s_2 \leq 0$ *and* $n \leq 3$.

In [6], Ozawa and Tsutaya proved the local well-posedness when $s_1 = s_2 = 0$ and $n \leq 3$. On the other hand, the Cauchy problem for the Zakharov system with data $(u, V, \partial_t V)|_{t=0} \in H^{s_1} \times H^{s_2} \times H^{s_2-1}$ is known to be locally well-posed, if $s_1 = 1/2$, $s_2 = 0$ for $n = 2, 3$, and $s_1 = 0$, $s_2 = -1/2$ for $n = 1$ (see [4]).

•Bourgain spaces

To prove Theorem 2, we need the two parameter family space $X^{s,b}$ which is introduced by Bourgain [1]. We refer to $X^{s,b}$ as the Bourgain space. The advantage of use of the Bourgain space is to reflect the nonlinear structure other than the degree of nonlinearities, although the space associated to the Strichartz estimate reflects only the degree of nonlinearities.

Now we give the definition of the Bourgain spaces. Let P be a real valued function on $\mathbb{R}^n$. We use $P(D_x)$ to denote the Fourier multiplier operator with the symbol P, thus we have $\Delta = -|D_x|^2$. Also we use $\widehat{F}$ to denote the space-time

Fourier transform of F. Then the Bourgain space $X^{s,b}_{P(D)}$ is defined as a space $e^{-itP(D_x)}\langle D_x\rangle^{-s}\langle D_t\rangle^{-b}L^2_{t,x}$ equipped with the norm

$$\|F\|_{X^{s,b}_{P(D)}} = \|\langle D_t\rangle^b\langle D_x\rangle^s e^{itP(D_x)}F\|_{L^2_{t,x}(\mathbb{R}^n\times\mathbb{R})}.$$

As well known, by the Plancherel theorem, we have

$$\|F\|_{X^{s,b}_{P(D)}} = \left\|\langle\xi\rangle^s\langle\tau+P(\xi)\rangle^b\widehat{F}\right\|_{L^2_{\tau,\xi}(\mathbb{R}^n\times\mathbb{R})}. \tag{7}$$

The key ingredient for the proof of Theorem 2 is the following bilinear estimate in the Bourgain space:

Theorem 3
Let $n \le 3$, $s_1, s_2 \le 0$ and $0 < b < 1/2$.
(i) If $s_1, s_2 > -1/2$, then we have

$$\|u\phi\|_{X^{s_1,-b}_{|D|^2}} \lesssim \|u\|_{X^{s_1,b}_{|D|^2}}\|\phi\|_{X^{s_2,b}_{\omega(D)}} \tag{8}$$

for some b sufficiently close to $1/2$, where we may replace ϕ with $\overline{\phi}$ in the left hand side of (8).
(ii) If $s_1 > -1/4$, then we have

$$\|\omega(D_x)|u|^2\|_{X^{s_2,-b}_{\omega(D)}} \lesssim \|u\|^2_{X^{s_1,b}_{|D|^2}} \tag{9}$$

for some b sufficiently close to $1/2$.

We see that, if $s_2 < -1/2$, then (8) fails, and if $s_1 < -b/2$, then (9) fails.

References

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