3つのδ関数を初期データに持つ非線形シュレーディンガー方程式の解の時間大域的評価について

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1 Introduction and Main Results

We consider the initial value problem of the nonlinear Schrödinger equation like

\[
\begin{align*}
\begin{cases}
i \partial_t u &= -\Delta u + \lambda \mathcal{N}(u) \\
u(0, x) &= \mu_{00} \delta_0 + \mu_{10} \delta_a + \mu_{01} \delta_b
\end{cases}
\end{align*}
\]  

(1.1)

where \((t, x) \in \mathbb{R} \times \mathbb{R}^n (n \geq 1), \partial_t = \partial / \partial t\) and \(\Delta = \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2 + \cdots + \partial^2 / \partial x_n^2\). The unknown variable \(u = u(t, x)\) takes a complex number. The nonlinearity \(\mathcal{N}(u)\) is of the gauge invariant power type given by

\[\mathcal{N}(u) = |u|^{p-1}u\quad\text{with}\quad1 < p < 1 + 2/n.\]

The nonlinear coefficient \(\lambda\) belongs to \(\mathbb{C}\) (the set of complex numbers). In particular, if \(\text{Im} \lambda < 0\), the nonlinear term causes dissipative effect. In the initial data, \(\delta_a\) denotes the well-known point mass measure supported at \(x = a \in \mathbb{R}^n\) and \(\mu_{jk}\) \((j, k = 0, 1)\) are complex numbers.

About (1.1), the speaker showed that

- If \(u(0, x) = \mu_0 \delta_0\), then \(u(t, x) = A(t) U(t) \delta_0\), where \(U(t) = \exp(it\Delta)\) and \(A(t)\) depends only on time variable \(t\). Note that \(A(t)\) blows up at \(t = T^* > 0\) if \(\text{Im} \lambda > 0\) and globally exists if \(\text{Im} \lambda \leq 0\).

- If \(u(0, x) = \mu_0 \delta_0 + \mu_1 \delta_a\), then \(u(t, x) = \sum_{j \in \mathbb{Z}} A_j(t) U(t) \delta_{ja}\). Note that, roughly speaking, \(A_j(t)\) blow up at \(t = T^* > 0\) if \(\text{Im} \lambda > 0\) and globally exist if \(\text{Im} \lambda \leq 0\).

Hence our present concern is to consider the triple δ-function case. If \(a = q b\) for some \(q \in \mathbb{Q}\) (\(\mathbb{Q}\) denotes the quotient number field), then the δ-functions are located at three points on the 1-dimensional lattice and (1.1) is solvable globally in time — the proof follows similarly to the double δ-function case. Therefore, in what follows, we restrict ourselves to observing the case \(a \neq q b\) for any \(q \in \mathbb{Q}\). Before stating the time local result, let us introduce several notations. The weighted sequence space \(\ell_2(\mathbb{Z}^2)\) is defined by

\[\ell_2^2(\mathbb{Z}^2) = \{\{A_{jk}\}_{j,k \in \mathbb{Z}}; \|\{A_{jk}\}_{j,k \in \mathbb{Z}}\|_{\ell_2^2(\mathbb{Z}^2)} < \infty\},\]
where \( \| \{ A_{jk} \}_{j,k \in \mathbb{Z}} \|^2_{\ell^2_{\alpha}(\mathbb{Z}^2)} = \sum_{j,k \in \mathbb{Z}} (1 + |j| + |k|^{2\alpha}) |A_{jk}|^2 \). For simplicity of the description, we often use \( \{ A_{jk} \} \) in place of \( \{ A_{jk} \}_{j,k \in \mathbb{Z}} \). Then the time local result is

**Theorem 1.1 (local result)** Let \( \lambda \in \mathbb{C} \) and \( 1 < \alpha < p \). Then, for some \( T > 0 \), there exists a unique solution to (1.1) described as

\[
(1.2) \quad u(t, x) = \sum_{j,k \in \mathbb{Z}} A_{jk}(t)U(t)\delta_{j\alpha+k\beta},
\]

where \( \{ A_{jk}(t) \} \in C([0, T]; \ell^2_{\alpha}(\mathbb{Z}^2)) \cap C^1((0, T]; \ell^2_{\alpha}(\mathbb{Z}^2)) \) with \( A_{jk}(0) = \mu_{jk} \) if \( (j, k) = (0, 0), (1, 0), (0, 1) \) and \( A_{jk}(0) = 0 \) otherwise.

**Remark 1.1.** The solution in Theorem 1.1 also causes the generation of new modes. Note that, for \( t \neq 0 \), \( U(-t)u(t) \) looks like a point mass measure supported at 2-dimensional lattice points if \( a \parallel b \), and densely distributed on the line along vector \( a \) if \( a \parallel b \) and \( a \neq qb \) for any \( q \in \mathbb{Q} \). Reading the proof of Theorem 1.1, we see that it is possible to construct a solution even when the initial data consists of infinitely many \( \delta \)-functions such as \( u(0, x) = \sum_{j,k \in \mathbb{Z}} \mu_{jk}\delta_{j\alpha+k\beta} \) with \( \{ \mu_{jk} \} \in \ell^2_{\alpha}(\mathbb{Z}^2) \) and \( \alpha > 1 \).

The sign of \( \text{Im}\lambda \) determines the blowing-up or global existence of the solution.

**Theorem 1.2 (blowing-up result)** Let \( \text{Im}\lambda > 0 \). Then, the solution in Theorem 1.1 blows up in positive finite time. Precisely speaking, \( \lim_{t \to T^*} \| \{ A_{jk}(t) \} \|_{\ell^2_{\alpha}(\mathbb{Z}^2)} = \infty \) for some \( T^* > 0 \).

As for the global existence, the difficulty largely depends on whether \( a \) and \( b \) are parallel or not, which does not arise in the single and double \( \delta \)-function case.

**Theorem 1.3 (global result)** (1) Let \( a \parallel b \). Then, if \( \text{Im}\lambda \leq 0 \), there exists a unique global solution to (1.1) described as in Theorem 1.1, where \( \{ A_{jk}(t) \} \in C([0, \infty); \ell^2_{\alpha}(\mathbb{Z}^2)) \cap C^1((0, \infty); \ell^2_{\alpha}(\mathbb{Z}^2)) \).

(2) Let \( a \parallel b \) and \( a \neq qb \) for any \( q \in \mathbb{Q} \). Then, if \( \text{Im}\lambda \leq 0 \) and additionally \( |\text{Re}\lambda| \leq \frac{2\sqrt{p}}{p-1} |\text{Im}\lambda| \), there exists a unique global solution to (1.1) described as in Theorem 1.1, where \( \{ A_{jk}(t) \} \in C([0, \infty); \ell^2_{\alpha}(\mathbb{Z}^2)) \cap C^1((0, \infty); \ell^2_{\alpha}(\mathbb{Z}^2)) \).

**Remark 1.2.** When \( a \parallel b \), the important matter is the equivalence of \( \| \{ (ja+k\beta)A_{jk} \} \|_{\ell^2_{\alpha}(\mathbb{Z}^2)} \) and \( \| \{ jA_{jk} \} \|_{\ell^2_{\alpha}(\mathbb{Z}^2)} + \| \{ kA_{jk} \} \|_{\ell^2_{\alpha}(\mathbb{Z}^2)} \). However, this is not the case if \( a \parallel b \). As for Theorem 1.3 (2), it is still open whether the additional condition \( |\text{Re}\lambda| \leq \frac{2\sqrt{p}}{p-1} |\text{Im}\lambda| \) is removed or not. In our proof, this condition will be applied to obtain the time global estimate of
\[ \| \{ A_{jk}(t) \} \|_{l_q^2(Z^2)} \] (This gives a rise to the desired estimate in \( l_q^2(Z^2) \)). The key to derive this estimate is Liskevich-Perelmuter’s inequality [5], i.e., if \( \text{Im} \lambda \leq 0 \) and \( |\text{Re} \lambda| \leq \frac{2\sqrt{p}}{p-1}|\text{Im} \lambda| \), then it follows that \( \text{Im} \left( \lambda (\mathcal{N}(v_1) - \mathcal{N}(v_2))(v_1 - v_2) \right) \leq 0 \).

We close this abstract by giving some more notations used in this talk. Let \( T = \mathbb{R}/2\pi \mathbb{Z} \) where \( \mathbb{Z} \) stands for the integer set. The quantity \( \| f \|_{L^q(T^2)} \) denotes \( \left( \int_{T^2} |f(\theta_1, \theta_2)|^q \, d\theta_1 d\theta_2 \right)^{1/q} \).

We next define the Besov space for periodic functions. Let \( s \) be the greatest integer not exceeding \( s \). Then, if \( s \) is not integer and \( 1 < q, r < \infty \), the Besov space \( B^s_{q,r}(T^2) \) is defined by

\[
B^s_{q,r}(T^2) = \{ f \in L^q(T^2); \, \| f \|_{B^s_{q,r}(T^2)} < \infty \},
\]

where

\[
\| f \|_{B^s_{q,r}(T^2)} = \| f \|_{L^q(T^2)} + \| f \|_{B^s_{q,r}} = \| f \|_{L^q(T^2)} + \left( \sum_{j,k} |\hat{f}_{jk}| \right)^{1/r}
\]

with \( h = (h_1, h_2) \) and \( d_N h(\theta_1, \theta_2) = \sum_{j=0}^{N} \binom{N}{j} (-1)^j h_1 \theta_1 + j h_2 \theta_2 \). We remark that, if \( 0 \leq \sigma \leq 1 \) and \( 1/q = \sigma/q_1 + (1-\sigma)/q_0 \) with \( 1 \leq q_1, q_0 \leq \infty \), then the Gagliardo-Nirenberg type inequality \( \| f \|_{B^{\sigma}_{q_1,q_1}(T^2)} \leq C \| f \|_{B^{\sigma}_{q_0,q_0}(T^2)} \| f \|_{L^{\sigma}(T^2)} \) follows from the above definition. We also note that \( \| f \|_{B^2_{2,2}(T^2)} \) is equivalent to

\[
\| f \|_{H^s(T^2)} = \left( \sum_{j,k \in \mathbb{Z}} (1 + |j| + |k|)^{2s} |C_{jk}|^2 \right)^{1/2},
\]

where \( C_{jk} \) is the Fourier coefficient of \( f \) given by \( (2\pi)^{-2} \int_{T^2} f(\theta_1, \theta_2) e^{-i(j\theta_1 + k\theta_2)} \, d\theta_1 d\theta_2 \).

Also, the inner product of \( f(\theta_1, \theta_2) \) and \( g(\theta_1, \theta_2) \in L^2(T^2) \) is defined by \( \langle f, g \rangle_{\theta_1, \theta_2} = \int_{T^2} f(\theta_1, \theta_2) g(\theta_1, \theta_2) \, d\theta_1 d\theta_2 \).

References


