Stability of standing waves for NLS equations with a delta function potential

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We consider the following nonlinear Schrödinger equation with a delta function potential:

$$i\partial_t u = -\partial_x^2 u + \gamma\delta(x)u + \alpha|u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1}$$

where γ, $\alpha \in \mathbb{R}$, $1 < p < \infty$, and $\delta(x)$ is the delta function at $x = 0$. The formal expression $-\partial_x^2 + \gamma\delta(x)$ in (1) is formulated as a linear operator A_γ or H_γ associated with a quadratic form a_γ on $H^1(\mathbb{R})$:

$$a_\gamma(u,v) = \int_{\mathbb{R}} \partial_x u(x)\overline{\partial_x v(x)}\,dx + \gamma u(0)\overline{v(0)}, \quad u, v \in H^1(\mathbb{R}).$$

Then, there exists a unique bounded linear operator $A_\gamma : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ such that

$$\langle A_\gamma u, v\rangle = \operatorname{Re} a_\gamma(u,v), \quad u, v \in H^1(\mathbb{R}).$$

Moreover, we define a linear operator H_γ in $L^2(\mathbb{R})$ by $H_\gamma v = -\partial_x^2 v$ for $v \in D(H_\gamma)$ with the domain

$$D(H_\gamma) = \{v \in H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R}) : \partial_x v(+0) - \partial_x v(-0) = \gamma v(0)\}.$$

Then, H_γ is a self-adjoint operator in $L^2(\mathbb{R})$, and satisfies

$$(H_\gamma u, v)_{L^2} = a_\gamma(u,v), \quad u, v \in D(H_\gamma).$$

The following spectral properties of H_γ are known: $\sigma_{\mathrm{ess}}(H_\gamma) = \sigma_{\mathrm{ac}}(H_\gamma) = [0,\infty)$, $\sigma_{\mathrm{sc}}(H_\gamma) = \emptyset$. If $\gamma \ge 0$, $\sigma_{\mathrm{p}}(H_\gamma) = \emptyset$. If $\gamma < 0$, $\sigma_{\mathrm{p}}(H_\gamma) = \{-\gamma^2/4\}$ with its positive normalized eigenfuction $(|\gamma|/2)^{1/2}e^{-|\gamma||x|/2}$ (see [1, Chapter I.3] for details).

In this talk, we study the structure and the orbital stability of standing wave solutions $e^{i\omega t}\varphi_\omega(x)$ for (1), where $\omega \in \mathbb{R}$ is a parameter, and $\varphi_\omega \in H^1(\mathbb{R})$ is a positive solution of the stationary problem:

$$A_\gamma \varphi + \omega\varphi + \alpha|\varphi|^{p-1}\varphi = 0 \quad \text{in } H^{-1}(\mathbb{R}). \tag{2}$$

The local well-posedness of the Cauchy problem for (1) in the energy space $H^1(\mathbb{R})$ follows from an abstract result in Cazenave [2]. Moreover, there is conservation of charge and energy, i.e.,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0), \quad \forall t \in [0, T^*),$$

where $u(t)$ is the solution of (1) with $u(0) = u_0 \in H^1(\mathbb{R})$, $T^* = T^*(u_0) \in (0, \infty]$ is the maximal existence time of $u(t)$, and E is defined by

$$\begin{aligned} E(v) &= \frac{1}{2}a_\gamma(v, v) + \frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2}\|\partial_x v\|_{L^2}^2 + \frac{\gamma}{2}|v(0)|^2 + \frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

(see Theorem 3.7.1 and Corollary 3.3.11 in [2]).

For the stability of standing waves for (1), the case where $\gamma < 0$ and $\alpha < 0$ was first studied by Goodman, Holmes and Weinstein [7] for the special case $p = 3$, and then by Fukuizumi, Ohta and Ozawa [6] for general case $1 < p < \infty$.

Theorem 1 ([6]) *Let $\gamma < 0$, $\alpha = -1$ and $1 < p < \infty$. If $\omega > \gamma^2/4$, the stationary problem (2) has a unique positive solution $\varphi_\omega \in H^1(\mathbb{R})$ given by*

$$\varphi_\omega(x) = \left(\frac{(p+1)\omega}{2}\right)^{1/(p-1)} \left\{\cosh\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + b(\omega)\right)\right\}^{-2/(p-1)} \tag{3}$$

for $x \in \mathbb{R}$, where $b(\omega) = \tanh^{-1}(-\frac{\gamma}{2\sqrt{\omega}})$. If $1 < p \le 5$, the standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) is stable for any $\omega \in (\gamma^2/4, \infty)$. If $p > 5$, there exists $\omega^* = \omega^*(\gamma, p) \in (\gamma^2/4, \infty)$ such that $e^{i\omega t}\varphi_\omega$ is stable for any $\omega \in (\gamma^2/4, \omega^*)$, and is unstable for any $\omega \in (\omega^*, \infty)$.

Remark that for the case where $\gamma = 0$ and $\alpha < 0$, the standing wave solution $e^{i\omega t}\varphi_\omega$ is stable for any $\omega \in (0, \infty)$ if $1 < p < 5$, and is unstable for any $\omega \in (0, \infty)$ if $p \ge 5$. Fukuizumi and Jeanjean [5] studies the case where $\gamma > 0$ and $\alpha < 0$. Note that the expression (3) of φ_ω is valid for the case where $\gamma > 0$ and $\alpha = -1$ (see also [4] for the expression of φ_ω). The stability of $e^{i\omega t}\varphi_\omega$ is determined by the sign of the derivative of the function

$$\omega \mapsto \|\varphi_\omega\|_{L^2}^2 = C_p \omega^{\frac{5-p}{2(p-1)}} \int_{b(\omega)}^{\infty} (\cosh y)^{-4/(p-1)}\, dy,$$

where C_p is a positive constant depending only on p. Moreover, the fact that φ_ω is a minimizer of the minimization problem

$$\inf\{S_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\},\ I_\omega(v) = 0\}$$

plays an important role in the proof of Theorem 1. Here, we put

$$S_\omega(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 + \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1},$$
$$I_\omega(v) = \|\partial_x v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 + \gamma|v(0)|^2 - \|v\|_{L^{p+1}}^{p+1}.$$

Next, we consider the case where $\gamma < 0$ and $\alpha > 0$ (attractive potential and repulsive nonlinearity).

Theorem 2 *Let $\gamma < 0$, $\alpha = 1$ and $1 < p < \infty$. If $0 < \omega < \gamma^2/4$, the stationary problem* (2) *has a unique positive solution $\varphi_\omega \in H^1(\mathbb{R})$ given by*

$$\varphi_\omega(x) = \left(\frac{(p+1)\omega}{2}\right)^{1/(p-1)} \left\{\sinh\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + c(\omega)\right)\right\}^{-2/(p-1)}$$

for $x \in \mathbb{R}$, where $c(\omega) = \tanh^{-1}(2\sqrt{\omega}/|\gamma|)$. The standing wave solution $e^{i\omega t}\varphi_\omega$ of (1) *is orbitally stable.*

Theorem 3 *Let $\gamma < 0$, $\alpha = 1$ and $\omega = 0$. If $1 < p < 5$, the stationary problem* (2) *has a unique positive solution $\varphi_0 \in H^1(\mathbb{R})$ given by*

$$\varphi_0(x) = \left(\frac{2(p+1)\gamma^2}{\{4 + (p-1)|\gamma||x|\}^2}\right)^{1/(p-1)}$$

for $x \in \mathbb{R}$. The stationary solution φ_0 of (1) *is orbitally stable. If $p \geq 5$, the stationary problem* (2) *has no nontrivial solutions in $H^1(\mathbb{R})$.*

The proof of Theorem 2 is based on the fact that φ_ω is characterized by a minimizer of the minimization problem

$$\inf\{S_\omega(v) : v \in H^1(\mathbb{R})\},$$

and the conservation of energy and charge (cf. Cazenave and Lions [3]). Theorem 3 is proved in a similar way.

References

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