楕円型方程式系の特異極限問題

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We consider

$$\begin{cases} -\Delta v = \mathbf{1}_{\Omega^+} - m, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \\ \beta v + \kappa = 0, & \text{on } \Gamma, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the smooth boundary $\partial\Omega$; $\partial/\partial n$ is the normal derivative on $\partial\Omega$; Ω^+ is an open set in Ω; $\Gamma = \partial\Omega^+ \subset \Omega$ is a C^2-curve embedded in Ω; κ is the curvature of Γ; $\beta > 0$ is a parameter; $m \in (0,1)$ is a constant; and $\mathbf{1}_{\Omega^+}$ denotes the characteristic function of Ω^+.

This problem comes from the following reaction-diffusion systems:

$$u_\tau = \epsilon^2 \Delta u + f(u) - \mu v, \qquad v_\tau = \Delta v + g(u,v), \tag{2}$$

where $u = u(y,\tau)$ and $v = v(y,\tau)$ are real-valued functions on $(y,\tau) \in \mathbb{R}^2 \times \mathbb{R}^+$; $\epsilon, \mu > 0$ are positive constants; $f \in C^1(\mathbb{R})$ is a function satisfying $f(i) = 0$, $f'(i) < 0$ $(i = 0,1)$, $f(a) = 0$, $f'(a) > 0$ with $a \in (0,1)$, and $f(s) = -W'(s)$ with $W \in C^2(\mathbb{R})$ being a double-equal-well potential satisfying

$$W(0) = W(1) = 0 < W(s) \quad {}^\forall s \in \mathbb{R}\backslash\{0,1\},$$
$$W''(0)W''(1) > 0;$$

and $g \in C^1(\mathbb{R}^2)$ is a function such that $g(1,0) = 1 - m > 0$, $g(-1,0) = -m < 0$. It follows $\int_0^1 f(s)\,ds = 0$.

A typical example of (f,g) is FitzHugh–Nagumo type: $f(s) = s(s-a)(1-s)$, $g(u,v) = u - \gamma v - m$ ($\gamma \geq 0$ is a nonnegative constant). The general case is referred to as the activator-inhibitor system.

The system (2) describes the reaction and the diffusion phenomena of substances. When the ratio of the diffusion constants, ϵ^2, is extremely small, very interesting stationary patterns, such as stripes or spots, often appear. As a mathematical approach to understand this pattern formation, we consider the limit $\epsilon \to 0$. Then usually the domain is divided into two regions and the remaining part becomes a thin layer. In some cases, the width of the internal transition layer approaches 0 in the limit, and the discontinuity surface inside the domain, which is called *sharp interface*, appears. Recently very fine layered patterns of (2) have attracted a great deal of attention. See [1, 2, 5, 6, 7, 8]. We consider this fine pattern which has the space scale of $\epsilon^{1/3}$ order. This is the unique scale that the order of the two driving forces of the sharp interface, the inhibitor v and the curvature of the sharp interface, balances; see [1, 5]. After rescaling $x = \frac{y}{\epsilon^{1/3}}$, $t = \epsilon^{4/3}\tau$, and $\varepsilon = \epsilon^{2/3}$, we obtain

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - \mu v), \\ \varepsilon^3 v_t = \Delta v + \varepsilon g(u, v). \end{cases} \tag{3}$$

We consider the stationary solutions of (3) subject to the homogeneous Neumann boundary condition:

$$\begin{cases} -\varepsilon^2 \Delta u = f(u) - \mu v, & \text{in } \Omega, \\ -\Delta v = \varepsilon g(u, v), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the smooth boundary $\partial\Omega$.

The reduced equation in the singular limit $\varepsilon \to 0$ becomes

$$\begin{cases} -\Delta v_1 = \mathbf{1}_{\Omega^+} - m, & \text{in } \Omega, \\ \dfrac{\partial v_1}{\partial n} = 0, & \text{on } \partial\Omega, \\ \frac{\mu}{\sigma} v_1 + \kappa = 0, & \text{on } \Gamma, \end{cases} \tag{RP}$$

where Ω^+ is an open set in Ω such that $\Gamma = \partial\Omega^+$ is a curve embedded in Ω; $\mathbf{1}_{\Omega^+}$ denotes the characteristic function of Ω^+. Here $u \to \mathbf{1}_{\Omega^+}$, $v/\varepsilon \to v_1$ as $\varepsilon \to 0$.

The essentially same equation as (RP) was obtained in [6] by using the matched expansion method. Once you have a "non-degenerate" solution of (RP) in some sense, you can find a layered solution for the singular perturbation problem (4) with $g(u, v) = u - m$ and $\mu = 1$. For the reduction from the parabolic system to the sharp interface model, see [12].

The problem (RP) has a variational structure associated with some energy functional. The direct method of calculus of variations implies the existence of global minimizers. This gives a solution of (RP). However it is usually difficult to know the profile and the non-degeneracy of the global minimizers. In this talk, we consider the problem to find a solution of (RP) which does not necessarily correspond to the global minimizers. The radially symmetric case for the related problems is studied in [3, 4, 6, 10, 11, 13]. We do not assume any symmetry of the domain.

In order to state the result, we define the Green's function and its harmonic part.

Definition 1 *For each $y \in \Omega$, let $G(x, y)$ be the solution to*

$$\begin{cases} -\Delta_x G(x,y) = \delta(x-y) - \dfrac{1}{|\Omega|}, & x \in \Omega, \\ \dfrac{\partial G}{\partial n_x}(x,y) = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega G(x,y)\,dx = 0. & \end{cases}$$

Set

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| + \frac{|x-y|^2}{4|\Omega|} + H(x,y), \quad x, y \in \Omega.$$

Then it is known that $H(x, y)$ is symmetric and harmonic in both x and y. Let $\mathcal{H}(x) = H(x, x)$.

We define the following two conditions.

(A1) $0 \in \Omega$ is a strict local minimum point of $\mathcal{H}$. More precisely, there exists a neighborhood U of 0 in Ω such that $\mathcal{H}(0) < \mathcal{H}(x)$ for all $x \in U \backslash \{0\}$.

(A2) $0 \in \Omega$ is a non-degenerate critical point of $\mathcal{H}$.

We denote by d_{H} Hausdorff metric

$$d_{\mathrm{H}}(K_1, K_2) = \max[\sup\{\mathrm{dist}(x, K_2)\,;\, x \in K_1\}, \sup\{\mathrm{dist}(y, K_1)\,;\, y \in K_2\}],$$

and $S_r(0) = \{x \in \mathbb{R}\,;\, |x| = r\}$. We have the following Theorem.

Theorem 1 *Assume that **(A1)** or **(A2)**. If $r_0 := \sqrt{\frac{m|\Omega|}{\pi}} < \mathrm{dist}(0, \partial\Omega)$, then there exists a constant $\beta_0 > 0$ such that (1) has a solution $\Gamma = \Gamma_\beta$, $v = v_\beta$, $\Omega^+ = \Omega^+_\beta$ for all $\beta < \beta_0$ satisfying $d_{\mathrm{H}}(\Gamma_\beta, S_{r_0}(0)) \to 0$ as $\beta \to 0$.*

In addition, we obtain the non-degeneracy result under **(A2)**.

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