分散型方程式の時空間評価式における比較原理とその応用

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This talk is based on a recent joint work with Michael Ruzhansky. We introduce a useful tool to derive new smoothing estimates from known ones. That is a comparison principle for solutions $u(t,x) = e^{itf(D_x)}\varphi(x)$ and $v(t,x) = e^{itg(D_x)}\varphi(x)$ to evolution equations with operators $f(D_x)$ and $g(D_x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$:

$$\begin{cases} (i\partial_t + f(D_x))\, u(t,x) = 0, \\ \hfill u(0,x) = \varphi(x) \end{cases} \quad \text{and} \quad \begin{cases} (i\partial_t + g(D_x))\, v(t,x) = 0, \\ \hfill v(0,x) = \varphi(x). \end{cases}$$

In the below, we write $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$, and $D_x = (D_1, D_2 \ldots, D_n)$ where D_j denotes D_{x_j} $(j = 1.2. \ldots, n)$. In the case $n = 1$, we neglect $x' = (x_2, \ldots, x_n)$ in natural way and just write $x = x_1$, $\xi = \xi_1$, and $D_x = D_1$. Similarly in the case $n = 2$, we use the notation $(x, y) = (x_1, x_2)$, $(\xi, \eta) = (\xi_1, \xi_2)$, and $(D_x, D_y) = (D_1, D_2)$.

Theorem 1. *Suppose $n = 1$. Let $f, g \in C^1(\mathbb{R})$ be real-valued and strictly monotone on the support of a function χ on $\mathbb{R}$. Let $\sigma, \tau \in C^0(\mathbb{R})$ be such that, for some $A > 0$, we have*

$$\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \le A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}$$

for all $\xi \in \operatorname{supp} \chi$ satisfying $f'(\xi) \neq 0$ and $g'(\xi) \neq 0$. Then we have

$$\|\chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \le A \|\chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

for all $x \in \mathbb{R}$. Consequently, for general $n \ge 1$ and for any function w on $\mathbb{R}^n$, we have

$$\begin{aligned} &\|w(x)\chi(D_j)\sigma(D_j)e^{itf(D_j)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \\ &\qquad \le A \|w(x)\chi(D_j)\tau(D_j)e^{itg(D_j)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)}, \end{aligned}$$

where $j = 1, 2, \ldots, n$.

Theorem 2. *Suppose $n = 2$. Let $f, g \in C^1(\mathbb{R}^2)$ be real-valued functions such that, for almost all $\eta \in \mathbb{R}$, $f(\xi, \eta)$ and $g(\xi, \eta)$ are strictly monotone in ξ on the support of a function χ on $\mathbb{R}^2$. Let $\sigma, \tau \in C^0(\mathbb{R}^2)$ be such that, for some $A > 0$, we have*

$$\frac{|\sigma(\xi, \eta)|}{|\partial f / \partial \xi(\xi, \eta)|^{1/2}} \le A \frac{|\tau(\xi, \eta)|}{|\partial g / \partial \xi(\xi, \eta)|^{1/2}}$$

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for all $(\xi, \eta) \in \operatorname{supp} \chi$ *satisfying* $\partial f/\partial \xi(\xi, \eta) \neq 0$ *and* $\partial g/\partial \xi(\xi, \eta) \neq 0$. *Then*

$$\left\|\chi(D_x, D_y)\sigma(D_x, D_y)e^{itf(D_x,D_y)}\varphi(x, y)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)} \\ \leq A\|\chi(D_x, D_y)\tau(D_x, D_y)e^{itg(D_x,D_y)}\varphi(x, y)\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)}$$

for all $x \in \mathbb{R}$. *Consequently, for general* $n \geq 2$ *and for any function* w *on* $\mathbb{R}^{n-1}$ *we have*

$$\|w(\check{x}_k)\chi(D_j, D_k)\sigma(D_j, D_k)e^{itf(D_j,D_k)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\ \leq A\|w(\check{x}_k)\chi(D_j, D_k)\tau(D_j, D_k)e^{itg(D_j,D_k)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},$$

where $j \neq k$ *and* $\check{x}_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$.

We also have a comparison result for radially symmetric case. In the below, we denote the set of the positive real numbers $(0, \infty)$ by $\mathbb{R}_+$.

Theorem 3. *Let* $f, g \in C^1(\mathbb{R}_+)$ *be real-valued and strictly monotone on the support of a function* χ *on* $\mathbb{R}_+$. *Let* $\sigma, \tau \in C^0(\mathbb{R}_+)$ *be such that, for some* $A > 0$, *we have*

$$\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leq A\frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

for all $\rho \in \operatorname{supp} \chi$ *satisfying* $f'(\rho) \neq 0$ *and* $g'(\rho) \neq 0$. *Then we have*

$$\|\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A\|\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

for all $x \in \mathbb{R}^n$. *Consequently, for any function* w *on* $\mathbb{R}^n$, *we have*

$$\|w(x)\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\ \leq A\|w(x)\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}.$$

Let us now give an important examples of the use of the comparison principle. We can conclude that many smoothing estimates for the Schrödinger type equations of different orders are equivalent to each other. Indeed, applying Theorem 1 in two directions, we immediately obtain that for $n = 1$ and $l, m > 0$, we have

$$\left\||D_x|^{(m-1)/2}e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} = \sqrt{\frac{l}{m}}\left\||D_x|^{(l-1)/2}e^{it|D_x|^l}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} \tag{1}$$

for every $x \in \mathbb{R}$, assuming that $\operatorname{supp} \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$. Applying Theorem 2, we similarly obtain that for $n = 2$ and $l, m > 0$, we have

$$\left\||D_y|^{(m-1)/2}e^{itD_x|D_y|^{m-1}}\varphi(x, y)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)} \\ = \left\||D_y|^{(l-1)/2}e^{itD_x|D_y|^{l-1}}\varphi(x, y)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)} \tag{2}$$

for every $x \in \mathbb{R}$. On the other hand, in the case $n = 1$, we have easily

$$\left\|e^{itD_x}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} = \|\varphi\|_{L^2(\mathbb{R}_x)} \quad \text{for all } x \in \mathbb{R}, \tag{3}$$

which is a straightforward result of the fact $e^{itD_x}\varphi(x) = \varphi(x + t)$. By using equality (3), we can estimate the right hand sides of equalities (1) and (2) with $l = 1$, and as a result, we have easily the following variety of pointwise estimates in low dimensions:

Proposition 1. *Suppose $n = 1$ and $m > 0$. Then we have*

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \le C \|\varphi\|_{L^2(\mathbb{R}_x)}$$

for all $x \in \mathbb{R}$. Suppose $n = 2$ and $m > 0$. Then we have

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \le C \|\varphi\|_{L^2\left(\mathbb{R}^2_{x,y}\right)}$$

for all $x \in \mathbb{R}$.

Proposition 1 with the special case $m = 2$ was shown by Kenig, Ponce and Vega [1, p.56] ($n = 1$) and Linares and Ponce [2, p.528] ($n = 2$). But we can say that these result, together with their generalization, are just corollaries of the elementary estimate (3).

By using the comparison principle for radially symmetric case, we have also another type of equivalence of smoothing estimates. In fact, by Theorem 3, we immediately obtain

$$\begin{aligned} \left\| |x|^{\beta-1} |D_x|^{\beta} e^{it|D_x|^2} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} &= \left\| |x|^{\beta-1} |D_x|^{\beta-1+m/2} e^{it|D_x|^m} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \\ &= \left\| |x|^{\alpha-m/2} |D_x|^{\alpha} e^{it|D_x|^m} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)}, \end{aligned}$$

where $m > 0$ and $\alpha = \beta - 1 + m/2$. On the other hand, we know the estimate

$$\left\| |x|^{\beta-1} |D_x|^{\beta} e^{it|D_x|^2} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^n_x)} \qquad (1 - n/2 < \beta < 1/2), \tag{4}$$

which was given by Sugimoto [3, Theorem 1.1]. Noticing that $1 - n/2 < \beta < 1/2$ is equivalent to $(m - n)/2 < \alpha < (m - 1)/2$, we have the following generalization of estimate (4):

Proposition 2. *Suppose $m > 0$ and $(m - n)/2 < \alpha < (m - 1)/2$. Then we have*

$$\left\| |x|^{\alpha-m/2} |D_x|^{\alpha} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

Further applications of the comparison principle will be given in the talk.

References

[1] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33–69.

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