Asymptotic stability of lattice solitons

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1 Introduction

In this article, we discuss stability of solitary waves of lattice equations which describe motion of infinite particles connected by a nonlinear spring:

$$(1) \qquad \begin{cases} p(t,n) = \dot{q}(t,n), \\ \dot{p}(t,n) = V'(q(t,n+1) - q(t,n)) - V'(q(t,n) - q(t,n-1)), \end{cases}$$

where $q(t,n)$ and $p(t,n)$ denote the displacement and momentum of the n-th particle at time t. A solitary wave solution is a single hump exponentialy localized solution which does not change its shape and speed for all the time. The existence of solitary waves is robust feature of FPU lattices due to a balance of nonlinearity and dispersion induced by discreteness of spatial variable. This was indicated by numerics [4] before Friesecke and Wattis [7] proved the existence of solitary waves by a variational method.

The FPU lattice equation was first studied by Fermi, Pasta and Ulam [5] who found recurrence phenomena for (1) with $V(r) = r^2/2 + r^3/6$ (α-FPU lattice) and $V(r) = r^2/2 + r^4/24$ (β-FPU lattice). A similar phenomena was observed by Zabusky and Kruskal [20] for KdV which is considered to be one of continuous limit of (1). They found *solitons* which consists of multiple solitary waves that are stable, collide each other elastically, can be back to the initial state after a certain time.

It is conjectured that recurrence phenomena does not hold rigorously except for integrable systems like KdV or the Toda lattice equation ((1) with $V(r) = e^{-r} - 1 + r$) and energy will be equally distributed to each Fourier mode after a long time if the total energy divided by a number of particles is large enough. That is, large solitary waves are expected to be metastable for finite FPU lattice. See [1, 19] and references therein for metastability results of finite lattices. Recently, Martel and Merle [11, 12] proved solitary

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waves of generalized KdV equations collide inelastically. Their results seem applicable to other non-integrable systems like BBM [13] and could be a clue to explain the conjecture.

As for infinite dimensional system (1), Friesecke and Pego [6] have proved stability of solitary waves under the hypothesis that solitary waves are linearly stable. They also proved the hypothesis for a small solitary waves. Since (1) lacks infinitesimal invariance of spatial variable, solitary waves cannot be characterized as a stationary point of a conserved quantity. Thus variational argument like [8] fails to explain stability of solitary waves of (1). Their idea is to build up a theory analogue to [17] using the fact that a major solitary wave outruns from other waves caused by perturbation. In fact, small solitary waves moves inherit linear stability property of KdV solitons in an exponentially weighted space.

One of our goal in this article is to prove every Toda 1-soliton solution is stable ([14]) without restriction of its amplitude. It is worth pointing out that stability of solitons does not necessary follows from integrability. For example, solitons of good Boussinesq equation are unstable if the traveling wave speed is sufficiently slow ([2]), and line solitons for the KP-I equation are unstable to long-wave transverse perturbations. The other goal is to explain that any linearly stable solitary waves of (3) are orbitally stable in the energy class ([15]).

2 Main Results

Let $r(t,n) = q(t,n+1) - q(t,n)$, $u = {}^t(r,p)$ and

$$H(u) = \sum_{n \in Z} \left(\frac{1}{2} p(t,n)^2 + V(r(t,n)) \right). \tag{2}$$

Eq. (1) can be translated into a Hamiltonian system

$$\frac{du}{dt} = J \nabla_u H(u), \tag{3}$$

where $J = \begin{pmatrix} 0 & e^{\partial} - 1 \\ 1 - e^{-\partial} & 0 \end{pmatrix}$ and $e^{\pm\partial}$ are shift operators such that $(e^{\pm\partial} f)(n) = f(n \pm 1)$. Every finite every solution of (3) satisfies a conservation law

$$E(u(t)) = E(u(0)) \quad \text{for every } t \geq 0. \tag{4}$$

A 1-soliton solution of the Toda lattice equation is given by

$$u_c(t,n) = (\partial_x q_c(n - ct + \delta), -c\partial_x q_c(n - ct + \delta)), \tag{5}$$

where $\delta \in \mathbb{R}$ is arbitrary, κ is a positive root of $c = \frac{\sinh \kappa}{\kappa} > 1$ and

$$q_c(x) = \log \frac{\cosh\{\kappa(x-1)\}}{\cosh \kappa x}. \tag{6}$$

Every 1-soliton solution of the Toda lattice equation is linearly stable.

Theorem 1 (with Pego). *Let $V(r) = e^{-r} - 1 + r$ and $u_c(t)$ be a 1-soliton solution defined by (5) and (6). Then there exist $K > 0$ and $a \in (0, 2\kappa)$ such that if a solution $w(t)$ of $w(t)$ of*

$$\partial_t w = JH''(u_c(t))w. \tag{LFPU}$$

satisfies

$$\langle w(s), J^{-1}\dot{u}_c(s)\rangle = \langle w(s), J^{-1}\partial_c u_c(s)\rangle = 0, \tag{SO}$$

$$\|e^{a(n-ct)}w(t)\|_{l^2} \le Ke^{-\beta(t-s)}\|e^{a(n-cs)}w(s)\|_{l^2} \quad \text{for } \forall\ t \ge s. \tag{L}$$

Remark 1. *(SO) is a secular term condition. If $w(t)$ satisfies (SO), then $w(t)$ does not include neutral modes $\dot{u}_c$ and $\partial_c u_c$.*

Remark 2. *(L) was proved for small solitary waves by Friesecke-Pego '04 for general nonlinearities.*

Theorem 2 (Stability in energy space). *Let $u(t)$ be a solution to (FPU) with $u(0) = u_{c_0}(0) + v_0$. Suppose (L). Then for $\forall \varepsilon > 0$, $\exists\ \delta > 0$ satisfying the following: If $\|v_0\|_{l^2} < \delta$, $\exists\ c_+ > 1$, $\sigma \in (1, c_+)$ and $x(t)$ such that*

$$\|u(t) - u_{c_0}(\cdot - x(t))\|_{l^2} < \varepsilon, \tag{7}$$

$$\lim_{t\to\infty} \left\|u(t) - u_{c_+}(\cdot - x(t))\right\|_{l^2(n\ge\sigma t)} = 0, \tag{8}$$

$$\sup_{t\in R} |\dot{x}(t) - c_0| = O(\|v_0\|_{l^2}), \quad \lim_{t\to\infty} \dot{x}(t) = c_+, \tag{9}$$

where $u_c(x) = {}^t(r_c(x), p_c(x))$.

Remark 3. *From [6], it remains unsolved whether solitary wave solutions are stable to a collision of solitary waves even if they are extremely small. In fact, small solitary waves does not belong to the exponentially weighted space $l^2 \cap l^2_a$ because they decay slowly as $n \to \pm\infty$.*

Remark 4. *Combining our method with Schneider and Wayne [18], Hoffman and Wayne [9] studied collision of counter-propagating solitary waves.*

3 Linearized Bäcklund transformation and stability of Toda lattice solitons

To prove linear stability, [6] compares spectral property of linearized FPU lattice equation with that of linearized KdV equation which is well known thanks to its integrability ([16]). In this section, we use algebraic structure Toda lattice equation, which is also integrable.

The Toda lattice admits a Bäcklund transformation determined by the equations

$$\begin{cases} \dot{q}_n + e^{-(q'_n - q_n)} + e^{-(q_n - q'_{n-1})} = \alpha, \\ \dot{q}'_n + e^{-(q'_n - q_n)} + e^{-(q_{n+1} - q'_n)} = \alpha, \end{cases} \tag{10}$$

where α is a constant. Presuming (10) holds, if $q(t) = q_n(t)$ is a solution to (1), then $q'(t,n) = q'_n(t)$ is necessarily a solution to (1) and vice-versa. In particular, the Bäcklund transformation connects the zero solution to 1-solitons: if $q'(t,n) \equiv 0$ and $\alpha = 2\cosh\kappa$, then

$$\dot{q}_n(t) + e^{q_n(t)} + e^{-q_n(t)} = 2\cosh\kappa, \tag{11}$$

$$e^{-q_{n+1}(t)} + e^{q_n(t)} = 2\cosh\kappa, \tag{12}$$

whence $q_n(t) = q_c(n - ct + \delta)$ and $\kappa > 0$ and $x \in \mathbb{R}$ are arbitrary constants independent of t.

Let us now linearize (10) around $q = q_c(n - ct)$ and $q' = 0$. This yields

$$p(t) + e^{q_c(t)}(q(t) - q'(t)) - e^{-q_c(t)}(q(t) - e^{-\partial}q'(t)) = 0, \tag{13}$$

$$p'(t) + e^{q_c(t)}(q(t) - q'(t)) - e^{\partial}e^{-q_c(t)}(q(t) - e^{-\partial}q'(t)) = 0. \tag{14}$$

Let $X_t = \{(q,p) \in l^2_a \times l^2_a : u = ((e^{\partial} - 1)q, p) \text{satisfies (SO)}\}$. This is a subspace corresponding to states u satisfying the secular term condition. A linearized Bäcklund transformation defines an isomorphism that connects solutions of (LFPU) in X_t and solutions of

$$\frac{dw}{dt} = Jw. \tag{15}$$

Propostion 3. *Suppose* $0 < a < 2\kappa$. *Let* $t \in \mathbb{R}$. *For every* $(q,p) \in X_t$, *there exists a unique* $(q',p') \in l^2_a \times l^2_a$ *satisfying* (13)–(14). *Furthermore, the map* $(q,p) \mapsto (q',p')$ *defines an isomorphism* $\Phi(t) \colon X_t \to l^2_a \times l^2_a$ *and The map* $\Phi_c(t)$ *and its inverse are uniformly bounded for*

$$\sup_{t \in R} \left(\|\Phi(t)\|_{B(X_t, l^2_a \times l^2_a)} + \|\Phi(t)^{-1}\|_{B(l^2_a \times l^2_a, X_t)} \right) < \infty.$$

Theorem 1 follows from Proposition 3 and a decay estimate corresponding to (L) for the linearized system of (3) around 0.

Lemma 4. *Let $a > 0$, $c > 1$ and $\beta = ca - 2\sinh(a/2)$. Then there exists a positive constant K such that for any solution w of (15) satisfy*

$$\|e^{a(\cdot - ct)}w(t)\|_{l^2} \le Ke^{-\beta(t-s)}\|e^{a(\cdot - cs)}w(s)\|_{l^2}.$$

Lemma 4 follows immediately by computing the Fourier transform of w.

Finally, we will give a sketch of the proof of Proposition 3.

Sketch of the proof. Eqs. (13) and (14) can be rewritten as

$$\begin{cases} C(t)q' = p + (e^{q_c(t)} - e^{-q_c(t)})q, \\ p' = (e^{q_c(t)} - e^{\partial}e^{-q_c(t)}e^{-\partial})q' - \hat{C}(t)q, \end{cases} \tag{16}$$

where $C(t) = e^{q_c(t)} - e^{-q_c(t)}e^{-\partial}$ and $\hat{C}(t) = e^{q_c(t)} - e^{\partial}e^{-q_c(t)}$ (formally $\hat{C}(t) = C(t)^*$). Thus to prove bijectivity of $\Phi(t)$, it suffices to prove the following.

Lemma 5. *Let $-2\kappa < a < 2\kappa$ and $t \in \mathbb{R}$. Let $C(t)$ be an operator on l^2_a, with adjoint $C^*(t) = e^{q_c(t)} - e^{\partial}e^{-q_c(t)}$ acting on $l^2_{-a} = (l^2_a)^*$. Then $C(t)$ is Fredholm,* $\ker C(t) = \{0\}$, $\ker C^*(t) = \operatorname{span}\{\dot{q}_c(t)\}$ *and* $\operatorname{Range} C(t)^* = l^2_{-a}$.

Lemma 6.

$$p + (e^{q_c(t)} - e^{-q_c(t)})q \perp \ker C(t)^*.$$

□

Remark 5. *Friesecke and Pego studied the spectrum property of a differential-difference operator of the second order to prove* (L) *for small solitary waves. Here, we limit ourselves to the Toda lattice equation and reduce the problem of the property of the first order difference operator $C(t)$. Our method is a kind of Darboux transformation, which is quite useful to investigate spectrum of linearized equation (see [16, 3]).*

4 Stability in the energy space

In this section, we discuss stability of solitary wave solutions in $l^2 \times l^2$. We will prove Theorem 2 using the fact that the main solitary wave outruns from perturbed waves. The idea was first used by Pego and Weinstein [17] to prove asymptotic stability of 1-soliton solutions of KdV equation in an exponentially weighted space. Later Friesecke and Pego

[6] applied the same idea to the FPU lattice equation. Recently, Martel and Merle [10] proved asymptotic stability of KdV 1-soliton in $H^1(\mathbb{R})$ by using a virial type estimate. Unfortunately, a viral type estimate may not be true around solitary wave solutions of the FPU lattice equation. In fact, [10] used orbital stability result to prove a virial type estimate. Unlike KdV equation, we cannot prove orbital stability independently by using conservation laws of the FPU lattice equation. Rather, we need to prove orbital and asymptotic stability at the same time.

Let us decompose a solution of (3) into a sum of the main solitary wave part and remainder part

$$u(t,n) = u_{c(t)}(n - x(t)) + v(t,n), \tag{17}$$

where $c(t)$ and $x(t)$ are speed and phase of the main solitary wave. Substituting (17) into (3), we have

$$\frac{dv}{dt} = JH''(u_{c(t)}(\gamma(t)))v(t) + l_1(t) + N_1(t), \tag{18}$$

where

$$\begin{aligned} l_1(t) =& - \dot{c}(t)\partial_c u_{c(t)}(\gamma(t)) - \frac{\dot{x}(t) - c(t)}{c(t)}\dot{u}_{c(t)}(\gamma(t)), \\ N_1(t) =& J\left\{H'(u_{c(t)}(\gamma(t)) + v(t)) - H'(u_{c(t)}(\gamma(t))) - H''(u_{c(t)}(\gamma(t)))v(t)\right\}. \end{aligned}$$

Now, we impose the following secular term condition for v and v_2.

$$\langle v, J^{-1}\dot{u}_c(\gamma)\rangle = 0, \tag{19}$$

$$\langle v_2, J^{-1}\partial_c u_c(\gamma)\rangle = 0. \tag{20}$$

Differentiating (19) and (20) with respect to t, we have modulation equations on $c(t)$ and $x(t)$.

Lemma 7. *Let $u(t)$ be a solution to* (3) *and $v_1(t)$ be a solution to* (21)*. Suppose that c and γ are C^1-functions satisfying* (19) *and* (20) *on* $[0,T]$ *and* $\inf_{t\in[0,T]} c(t) > 1$*. Then it holds for $t \in [0,T]$ that*

$$\begin{aligned} &\dot{c}(t) = O(\|v_1(t)\|_{W(t)}^2 + \|v_2(t)\|_{X(t)}^2), \\ &\dot{x}(t) - c(t) = O(\|v_1(t)\|_{W(t)} + (\|v(t)\|_{l^2} + \|v_1(t)\|_{l^2})\|v_2(t)\|_{X(t)}), \end{aligned}$$

where $\|u\|_{W(t)} = \left(\sum_{n\in Z} e^{-\kappa(c(t))|n-x(t)|}|u(n)|^2\right)^{1/2}$, $\|u\|_{X(t)} = e^{-ax(t)}\|u\|_{l^2_a}$ *and a is a constant satisfying* $0 < a \le \inf_{t\in[0,T]}\kappa(c(t))$.

Using the conservation law (4), we have the following.

Lemma 8. *Suppose* (19). *Then* $\|v(t)\|_{l^2}^2 \lesssim |c(t)-c_0| + \|v_0\|_{l^2}$.

In view of Lemmas 7 and 8, we find that a solitary wave solution is orbitally stable if $\|v\|_{W(t)}$ is square integrable with respect to t.

The remainder term $v(t)$ includes small solitary waves and radiation part of the solution and they move more slowly compared with the main wave. Now, we will extract a roughly decaying part of $v(t)$ as $n \to \pm\infty$, which is a hazard to apply a semigroup estimate (L). Let $v_1(t)$ be a solution of

$$\partial_t v_1 = JH'(v_1), \quad v_1(0) = v_0, \tag{21}$$

and let $v_2(t) = v(t) - v_1(t)$. Here v_0 is a perturbation to a solitary wave solution at $t = 0$. We have a virial type estimate for small v_1.

Lemma 9. *Let $v_1(t)$ be a solution to* (21). *Let $c_1 > 1$ and $\tilde{x}(t)$ be a C^1-function satisfying $\inf_{t\in R} \tilde{x}_t \ge c_1$. Then there exist positive numbers a_0 and δ_3 such that if $a \in (0, a_0)$ and $\|v_0\|_{l^2} \le \delta_3$,*

$$\|\psi_a(t)^{1/2} v_1(t)\|_{l^2}^2 + \int_0^t \|\tilde{\psi}_a(t) v_1(s)\|_{l^2}^2 ds \lesssim \|\psi_a(0)^{1/2} v_0\|_{l^2}^2,$$

where $\psi_a(t,x) = 1 + \tanh a(x - \tilde{x}(t))$ and $\tilde{\psi}_a(t,x) = a^{1/2} \operatorname{sech} a(x - \tilde{x}(t))$.

The above lemma use the fact that *the maximum speed* of waves of v_1 is at most $1 + o(1)$. Lemma 9 implies

$$\infty_0^\infty \|v_1\|_{W(t)}^2 dt \lesssim \|v_0\|_{l^2}^2. \tag{22}$$

Subtracting (21) from (18), we obtain

$$\begin{cases} \dfrac{dv_2}{dt} = JH''(u_{c(t)}(\gamma(t)))v_2 + l_1(t) + N_2(t), \\ v_2(0) = u_{c_0}(\tau_0) - u_{c(0)}(\gamma(0)), \end{cases} \tag{23}$$

where $N_2(t) = N_1(t) - JH'(v_1(t)) + JH''(u_{c(t)}(\gamma(t)))v_1$. Note that initial data of v_2 is exponentially localized and so is the nonlinear term N_2. In fact,

$$N_2 \simeq \text{a linear combination of } u_c v_1,\ v_2^2,\ v_1 v_2.$$

Hence by applying Theorem 1 or [6] to (23), we obtain

$$\|v_2(t)\|_{X(t)} + \left(\int_0^\infty \|v_2(t)\|_{X(t)}^2 dt\right)^{1/2} \lesssim \|v_0\|_{l^2} + \left(\int_0^\infty \|v_1(t)\|_{W(t)}^2 dt\right)^{1/2}.$$

Thus we have

$$\|v_2(t)\|_{L_t^2(W(t))} \lesssim \|v_2(t)\|_{L_t^2(X(t))} \lesssim \|v_0\|_{l^2}. \tag{24}$$

Combining (22) and (24) with Lemmas 9 and 8, we obtain (7) and (9).

Once we prove (7), we can prove virial identity around the main solitary wave solution as in [10] which implies (8).

Appendix

Proof of Lemma 8. By (19), we have $\langle H'(u_c), v\rangle = \langle J^{-1}\dot{u}_c, v\rangle = 0$. Since $H(u(t))$ is independent of t,

$$\begin{aligned}
\delta H :=& H(u(0)) - H(u_{c_0}) \\
=& H(u_{c(t)}) + \langle H'(u_{c(t)}), v\rangle + \frac{1}{2}\langle H''(u_c)v, v\rangle - H(u_{c_0}) + O(\|v\|_{l^2}^3) \\
\geq& \frac{1}{4}\|v(t)\|_{l^2}^2 + O(|c(t)-c_0| + \|v(t)\|_{l^2}^3).
\end{aligned}$$

Since $\delta H = O(\|v_0\|_{l^2})$, we have Lemma 8. □

Proof of Lemma 7. Differentiating (19) with respect to t and substituting (18) into the resulting equation, we have

$$\begin{aligned}
&\frac{d}{dt}\langle v, J^{-1}\dot{u}_c(\gamma)\rangle \\
=&\langle \dot{v}, J^{-1}\dot{u}_c(\gamma)\rangle + \frac{\dot{x}}{c}\langle v, J^{-1}\ddot{u}_c(\gamma)\rangle + \dot{c}\langle v, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle \\
=&\langle JH''(u_c(\gamma))v, J^{-1}\dot{u}_c(\gamma))\rangle + \langle v, J^{-1}\ddot{u}_c(\gamma)\rangle \\
&+ \langle l_1 + N_1, J^{-1}\dot{u}_c(\gamma)\rangle + \left(\frac{\dot{x}}{c} - 1\right)\langle v, J^{-1}\ddot{u}_c(\gamma)\rangle + \dot{c}\langle v, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle \\
=&0.
\end{aligned}$$

Substituting $\ddot{u}_c = JH''(u_c)\dot{u}_c$ and $J^* = -J$ into the above, we have

$$\dot{c}\left\{\frac{d}{dc}H(u_c) - \langle v, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle\right\} - \left(\frac{\dot{x}}{c} - 1\right)\langle v, J^{-1}\ddot{u}_c(\gamma)\rangle = \langle N_1, J^{-1}\dot{u}_c(\gamma)\rangle. \tag{25}$$

Differentiating (20) with respect to t, we have

$$\begin{aligned}
&\frac{d}{dt}\langle v_2, J^{-1}\partial_c u_c(\gamma)\rangle \\
=&\langle \dot{v}_2, J^{-1}\partial_c u_c(\gamma)\rangle + \frac{\dot{x}}{c}\langle v_2, J^{-1}\partial_c \dot{u}_c(\gamma)\rangle + \dot{c}\langle v_2, J^{-1}\partial_c^2 u_c(\gamma)\rangle \\
=&\langle JH''(u_c(\gamma))v_2, J^{-1}\partial_c u_c(\gamma))\rangle + \langle v_2, J^{-1}\partial_c \dot{u}_c(\gamma)\rangle \\
&+ \langle l_1 + N_2, J^{-1}\partial_c u_c(\gamma)\rangle + \left(\frac{\dot{x}}{c} - 1\right)\langle v_2, J^{-1}\partial_c \dot{u}_c(\gamma)\rangle + \dot{c}\langle v_2, J^{-1}\partial_c^2 u_c(\gamma)\rangle \\
=&0.
\end{aligned}$$

Substituting $\partial_c \dot{u}_c = JH''(u_c)\partial_c u_c$ into the above, we obtain

(26)
$$\begin{aligned}
&\left(\frac{\dot{x}}{c} - 1\right)\left\{\frac{d}{dc}H(u_c) + \langle v_2, J^{-1}\partial_c \dot{u}_c(\gamma)\rangle\right\} + \dot{c}\left\{\langle \partial_c u_c, J^{-1}\partial_c u_c\rangle + \langle v_2, J^{-1}\partial_c^2 u_c(\gamma)\rangle\right\} \\
=&-\langle N_2, J^{-1}\partial_c u_c(\gamma)\rangle.
\end{aligned}$$

Since $|N_1(t)| \lesssim |v(t)|^2$ and $|J^{-1}\dot{u}_c(t,n)| \lesssim e^{-2\kappa(c)|n-x(t)|}$ as $n \to \infty$, we have

$$\langle N_1, J^{-1}\dot{u}_c(\gamma)\rangle = O(\|v(t)\|_{W(t)}^2).$$

Let $N_2(t) = \widetilde{N}_1(t) + \widetilde{N}_2(t) + \widetilde{N}_3(t)$, where

$$\begin{aligned}
\widetilde{N}_1(t) =& N_1(t) - JH'(v(t)) + Jv(t), \\
\widetilde{N}_2(t) =& JH'(v(t)) - JH'(v_1(t)) - Jv_2(t), \\
\widetilde{N}_3(t) =& J\left(H''(u_{c(t)}(\gamma(t))) - 1\right)v_1(t).
\end{aligned}$$

We put $G(v) := H'(v) - H'(0) - H''(0)v$ so that $JG(v)$ denotes a part of $\widetilde{N}_1(t)$ that does not interact with the solitary wave $u_c(\gamma)$. Since $|u_c(t,n)| \lesssim e^{-2\kappa(c)|n-x(t)|}$ and $a \le \inf_{t\in[0,T]} \kappa(c(t))$, we have $\|u_{c(t)}v^2\|_{X(t)} \lesssim \|v\|_{W(t)}^2$. Hence by the definition of $\widetilde{N}_1$ and $\widetilde{N}_2$,

(27) $$\|\widetilde{N}_1(t)\|_{X(t)} = \|N_1(t) - JG(v(t))\|_{X(t)} \lesssim \|v(t)\|_{W(t)}^2,$$

(28) $$\begin{aligned}
\|\widetilde{N}_2(t)\|_{X(t)} =&\|JG(v(t)) - JG(v_1(t))\|_{X(t)} \\
\lesssim&(\|v(t)\|_{l^\infty} + \|v_1(t)\|_{l^\infty})\|v_2(t)\|_{X(t)}.
\end{aligned}$$

We see from (5), (6) or [6] that $H''(u_c) - 1$ decays like $e^{-2\kappa|n-x(t)|}$ as $n \to \pm\infty$ and for $a \in (0, \kappa(c(t)))$,

(29) $$\|\widetilde{N}_3(t)\|_{X(t)} \lesssim \|v_1(t)\|_{W(t)}.$$

Let $\|u\|_{X(t)^*} = e^{ax(t)}\|u\|_{l^2_{-a}}$ and $\|u\|_{W(t)^*} = (\sum_{n\in Z} e^{\kappa(c(t))|n-x(t)|}|u(n)|^2)^{1/2}$. In view of (25), (26) and the fact that

$$\sup_{t\in[0,T]} \left(\|J^{-1}\ddot{u}_{c(t)}(\gamma(t))\|_{W(t)^*} + \|J^{-1}\partial_c\dot{u}_{c(t)}(\gamma(t))\|_{W(t)^*}\right) < \infty,$$
$$\sup_{t\in[0,T]} \left(\|J^{-1}\partial_c\dot{u}_{c(t)}(\gamma(t))\|_{X(t)^*} + \|J^{-1}\partial_c^2 u_{c(t)}(\gamma(t))\|_{X(t)^*}\right) < \infty,$$

we have

$$\mathcal{A}(t)\begin{pmatrix} \dot{c}(t) \\ \dot{x}(t) - c(t)\end{pmatrix} = \begin{pmatrix} O(\|v(t)\|^2_{W(t)}) \\ O(\|v_1(t)\|_{W(t)} + (\|v(t)\|_{l^2} + \|v_1(t)\|_{l^2})\|v_2(t)\|_{X(t)})\end{pmatrix},$$

where $\mathcal{A}(t) = \mathrm{diag}(dH(u_c)/dc, dH(u_c)/dc) + O(\|v_1(t)\|_{W(t)} + \|v_2(t)\|_{X(t)})$. We have thus proved Lemma 7. □

References

[1] D. Bambusi and A. Ponno, On metastability in FPU, Comm. Math. Phys. **264** (2006), 539–561.

[2] Bona J L and Sachs R L, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, *Comm. Math. Phys.* ***118*** *(1988), 15–29.*

[3] S. M. Chang, S. Gustafson, K. Nakanishi and T. P. Tsai*, Spectra of linearized operators for NLS solitary waves, SIAM J. Math. Anal. 39o (2007/08), 1070–1111.*

[4] Eilbeck J C and Flesch R*, Calculation of families of solitary waves on discrete lattices Phys. Lett. A* ***149*** *(1990), 200–202*

[5] E Fermi, J Pasta and S Ulam*, Studies of nonlinear problems, LASL Report LA-1940.*

[6] G. Friesecke and R.L. Pego*, Solitary waves on FPU lattices, Nonlinearity* ***12*** *(1999), 1601–1627; Nonlinearity* ***15*** *(2002), 1343–1359; Nonlinearity* ***17*** *(2004), 207–227; Nonlinearity* ***17*** *(2004), 229–251.*

[7] G. Friesecke and J. Wattis*, Existence theorem for solitary waves on lattices, Commun. Math. Phys.* ***161*** *(1994), 391–418.*

[8] M. Grillakis, J. Shatah and W. A. Strauss, *Stability Theory of solitary waves in the presence of symmetry I, J. Diff. Eq.* ***74*** *(1987), 160–197.*

[9] A. Hoffman and C. E. Wayne *Counter-propagating two-soliton solutions in the Fermi–Pasta–Ulam lattice, Nonlinearity* ***21*** *(2008), 2911–2947.*

[10] Y. Martel and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations, Arch. Ration. Mech. Anal.* ***157*** *(2001), no. 3, 219–254.*

[11] Y. Martel and F. Merle, *Description of two soliton collision for the quartic gKdV equation, Arxiv preprint arXiv:0709.2672 (2007).*

[12] Y. Martel and F. Merle, *Stability of two soliton collision for nonintegrable gKdV equations, Arxiv preprint arXiv:0709.2677, (2007).*

[13] Y. Martel, F. Merle and T. Mizumachi, *Description of the inelastic collision of two solitary waves for the BBM equation, Arxiv preprint arXiv:0803.4020 (2008).*

[14] T. Mizumachi and R.L. Pego, *Asymptotic stability of Toda lattice solitons, Nonlinearity 21 (2008), 2061–2071. .*

[15] T. Mizumachi *Asymptotic stability of lattice solitons in the energy space, Arxiv preprint arXiv:0711.2134 (2007).*

[16] R. L. Pego and M. I. Weinstein, *Eigenvalues and instabilities of solitary waves, Phil. Trans. Roy. Soc. London A, 340 (1992), 47–94.*

[17] R. L. Pego and M. I. Weinstein, *Asymptotic stability of solitary waves, Comm. Math. Phys., 164 (1994), 305–349.*

[18] G. Schneider and C. E. Wayne, *Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi-Pasta-Ulam model, International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 390–404, World Sci. Publ.*

[19] M. Toda, *Theory of nonlinear lattices (Second edition), Springer Series in Solid-State Sciences****20****, 1989.*

[20] N.J. Zabusky and M. D. Kruskal, *Interaction of solitons in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett.* ***15*** *(1965).*