On a regularity criterion for a weak solution to the Navier-Stokes equations

Fu Iwakawa
Department of Mathematics, Graduate School of Science and Technology
Kumamoto University

In this paper, we study a regularity of a solution to the Cauchy problem for the Navier-Stokes equations. For a vector-valued function $v = v(t,x) = (v_1(t,x), \ldots, v_n(t,x))$ and a real-valued function $p = p(t,x)$ defined for $(t,x) \in (0,T) \times \mathbf{R}^n$, $n \geq 2$ and $T > 0$, we consider the following Cauchy problem for the Navier-Stokes equations:

$$v_t - \Delta v + (v \cdot \nabla)v + \nabla p = 0 \quad \text{in} \quad (0,T) \times \mathbf{R}^n, \tag{1.1}$$

$$\nabla \cdot v \equiv \text{div } v = 0 \quad \text{in} \quad (0,T) \times \mathbf{R}^n, \tag{1.2}$$

$$v|_{t=0} = v_0 \quad \text{on} \quad \mathbf{R}^n, \tag{1.3}$$

where $v_0 = v_0(x) = (v_{0,1}(x), \cdots, v_{0,n}(x))$ is a given initial datum on $\mathbf{R}^n$. In terms of fluid mechanics, $v(t,x)$ and $p(t,x)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point of $(t,x) \in (0,T) \times \mathbf{R}^n$, respectively. The divergence free condition (1.2) implies the incompressibility of the fluid under consideration and the second term of (1.1) represents a viscosity of the fluid. The system of equations (1.1) and (1.2) describes a mathematical model for a dynamics of an incompressible viscous fluid in $\mathbf{R}^n$, called to be the Navier-Stokes equations, and has a long history as one of most important subject in the mathematical analysis.

Before starting our results, we introduce some function spaces. Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ functions $\phi = (\phi^1, \phi^2, \cdots, \phi^n)$ with compact support in $\mathbf{R}^n$ such that $\nabla \cdot \phi = 0$. L^r_σ is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r-norm $\| \cdot \|_r$; $(\cdot\, , \cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual function space of L^r-integrable vector-valued functions on $\mathbf{R}^n, 1 \leq r \leq \infty$. H^s_σ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the H^s-norm

$$\|\phi\|_{H^s} = \left(\sum_{|\alpha|=s} \|\nabla^\alpha \phi\|_2^2 \right)^{\frac{1}{2}},$$

where, for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$,

$$\nabla^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}}, \quad \nabla_i = \frac{\partial}{\partial x_i}, \quad i = 1, \cdots, n,$$

denote the partial differentiation with respect to $x = (x_1, \cdots, x_n)$.

Definition 1.1 (functions of bounded mean oscillation) *Let f be a locally integrable in $\mathbf{R}^n$, denoted by $f \in L^1_{loc}(\mathbf{R}^n)$. We say that f is bounded mean oscillation (abbreviated as BMO) if*

$$\|f\|_{BMO(\mathbf{R}^n)} = \sup_{B \subset \mathbf{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where the supremum ranges over all finite ball $B \subset \mathbf{R}^n$, $|B|$ is the n-dimensional Lebesgue measure of B, and f_B denote the mean value of f over B, namely $f_B = 1/|B| \ \int_B f(x)dx$.

The class of functions of BMO, modulo constants, is a Banach space with the norm $\| \cdot \|_{BMO(\mathbf{R}^n)}$ defined above.

The purpose of this paper is to give a condition of extension of solutions to (1.1) *and* (1.2).

Theorem 1.2 *Let $s > n/2 - 1$ and let $v_0 \in H^2_\sigma$. Suppose that v is a smooth solution of* (1.1), (1.2) *and* (1.3). *If*

$$\int_{\varepsilon_0}^T \|v(t)\|^2_{BMO(\mathbf{R}^n)} \, dt < \infty \quad \text{for} \quad 0 < \varepsilon_0 < T, \tag{1.4}$$

then v can be continued to a smooth solution for some $T' > T$.

We know by [4] that for any $v_0 \in H^s$, $s > n/s - 1$, there exist a positive number T depending on $\|v_0\|_{H^s}$ and a solution u of (1.1), (1.2) and (1.3) in the class

$$u \in C([0,T); H^s) \cap C^1(0,T; H^s) \cap C(0,T; H^{s+2}). \tag{1.5}$$

It becomes an interesting problem whether the solution $u(t)$ above is smooth beyond $t = T$. It is known in [5] that if

$$\int_0^T \|u(t)\|_r^k dt < +\infty \quad for \quad \frac{2}{k} + \frac{n}{r} = 1, \quad n < r \leq \infty,$$

then u can be continued to the solution in the class (1.5) above beyond $t = T$ (see also [8],[9]). We also have in [6] that Theorem 1.2 is valid. In this paper, we report on an alternative proof of Theorem 1.2.

We use a function space to prove our main theorem, Theorem 1.2.

Definition 1.3 (Hardy space) *The Hardy space consists of functions $f \in L^1(\mathbf{R}^n)$ such that*

$$\|f\|_{\mathcal{H}^1(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \sup_{r>0} |\phi_r * f(x)| dx$$

is finite, where $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ *for* $r > 0$ *and* ϕ *is a smooth function on* $\mathbf{R}^n$ *with compact support in an unit ball with center of the origin* $B_1(0) = \{x \in \mathbf{R}^n; |x| < 1\}$*.*

We know that the definition dose not depend on choice of a function ϕ*.*

We can refer to [3] in some properties of the Hardy space.

We have the decisive duality between the Hardy space and BMO, which plays an important role in the proof of our main theorem (refer to [3]).

Theorem 1.4 (Fefferman-Stein inequality) *It holds that* $(\mathcal{H}^1(\mathbf{R}^n))^* = BMO$*. Furthermore, there is a positive constant* C *depending only on* n *such that*

$$\left|\int_{\mathbf{R}^n} f(x)g(x)dx\right| \leq C\|f\|_{\mathcal{H}^1(\mathbf{R}^n)}\|g\|_{BMO(\mathbf{R}^n)}$$

holds for any $f \in \mathcal{H}^1(\mathbf{R}^n)$ *and* $g \in BMO(\mathbf{R}^n)$*.*

The following in equalities are used to estimate solutions of (1.1), (1.2) and (1.3) (see to [2]).

Lemma 1.5 (Sobolev inequality) *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain with smooth boundary* $\partial\Omega$ *and* $u \in W^{1,p}(\Omega)$ *with* $1 \leq p < n$*. Then* $u \in L^{\frac{np}{n-p}}(\Omega)$ *and we have the estimate*

$$\|u\|_{L^{\frac{np}{n-p}}} \leq \frac{np-p}{n-p}\|\nabla u\|_{L^p} + |\Omega|^{-\frac{1}{n}}\|u\|_{L^p}$$

Lemma 1.6 (Hardy-Littlewood maximal inequality) *For* $f \in L^1(\mathbf{R}^n)$*, we define the maximal function* $M(f)$ *by*

$$M(f)(x) = \sup_{r>0}\frac{1}{|B_r|}\int_{B_r(x)}|f(y)|dy.$$

It holds for any $f \in L^p(\mathbf{R}^n)$*,* $p > 1$*, that*

$$\|Mf\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)},$$

where the positive constant C *depends only on* p *and* n*.*

Now we state a priori estimate which plays a fundamental role to prove our main theorem, Theorem 1.2.

Theorem 1.7 *Let* $v_0 \in W^{2,2}(\mathbf{R}^n) \cap L^2_\sigma$ *such that* $\nabla\cdot v_0 = 0$*. Suppose that* $v \in L^2((0,T)\times\mathbf{R}^n)$ *is a smooth solution of* (1.1), (1.2) *and* (1.3) *and satisfies*

$$\int_0^T \|v(t)\|^2_{BMO(\mathbf{R}^n)}dt < \infty. \tag{1.6}$$

Then it holds that

$$\max_{0\le t\le T}\|v_t(t)\|_2^2 \quad + \quad \int_0^T \|\nabla v_t(t)\|_2^2 dt \le C_1, \tag{1.7}$$

$$\max_{0\le t\le T}(\|v(t)\|_2^2 \quad + \quad \|\nabla v(t)\|_2^2) \le C_2, \tag{1.8}$$

where

$$\begin{aligned} C_1 &= \|v_t(0)\|_2^2 \ \exp\left(C\int_0^T \|v(t)\|^2_{BMO(\mathbf{R}^n)}dt\right) \\ &\quad +C\left(\|v_t(0)\|_2^2 + \sup_{0\le t\le T}\|v_t\|_2^2\int_0^T \|v(t)\|^2_{BMO(\mathbf{R}^n)}dt\right), \\ C_2 &= \|v_0\|_2^2 + C\|v_0\|_2^4 + \|\nabla v(0)\|_2^2, \end{aligned}$$

where the positive constant C depends only on n.

To show the validity of Theorem 1.7, we derive the integral identities available for smooth solutions to (1.1), (1.2) and (1.3).

Lemma 1.8 (Energy equalities) *Let v be a smooth solution of* (1.1), (1.2) *and* (1.3) *Suppose that derivatives of the form $\nabla_k\nabla_l v_t$ and of all lower orders are L^2-integrable on $(0,t)\times\mathbf{R}^n$. Then the following identities hold:*

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \|\nabla v\|_2^2 = 0, \tag{1.9}$$

$$\frac{1}{2}\frac{d}{dt}\|v_t\|_2^2 + \|\nabla v_t\|_2^2 + (\,(v_t\cdot\nabla)v, v_t) = 0, \tag{1.10}$$

$$\|v_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 + (\,(v\cdot\nabla)v, v_t) = 0. \tag{1.11}$$

Moreover, we need a priori estimates for higher derivatives of a solution.

Theorem 1.9 (higher derivative estimate) *Under the same assumptions as in Theorem 1.7, it holds that*

$$\begin{aligned} &\max_{0\le t\le T}\|\nabla^\alpha v(t)\|_2^2 + \int_0^T \left\|\nabla^{\alpha+1}v(t)\right\|_2^2 dt \\ \le\ & C\left(\|\nabla^\alpha v(0)\|_2^2 + \int_0^T \|v(t)\|^2_{BMO(\mathbf{R}^n)}\,dt\right)\exp\left(C\int_0^T \|v(t)\|^2_{BMO(\mathbf{R}^n)}dt\right), \end{aligned}$$

where the positive constant $C = C(n, r, \alpha)$.

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