BEHAVIOUR OF SOLUTIONS TO
THE LINEAR WAVE EQUATIONS IN EXTERIOR DOMAINS

Shoko Okamura
Department of Mathematics, Graduate School of Science and Technology
Kumamoto University

1. Introduction.

Let Ω be an exterior domain in $\mathbf{R}^2$ with a compact C^2-boundary $\partial\Omega$. Without loss of generality we may assume $(0,0) \notin \overline{\Omega}$. In this paper, we will consider the Cauchy-Dirichlet problem for the wave equation. For a function $u = u(t,x)$ defined for $(t,x) \in (0,\infty) \times \Omega$, we study the following initial-boundary value problem for the wave equation:

$$u_{tt}(t,x) - \Delta u(t,x) = 0 \quad \text{in} \quad (0,\infty) \times \Omega, \tag{1.1}$$

$$u(t,x) = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega, \tag{1.2}$$

$$u(t,x) = u_0,\ u_t(t,x) = u_1 \quad \text{on} \quad \{t = 0\} \times \Omega. \tag{1.3}$$

Throughout this paper, we use the usual notations. For $f, g \in L^2(\Omega)$,

$$(f,g) = \int_\Omega f(x)g(x)dx, \qquad \|f\|_{L^2(\Omega)} = \sqrt{(f,f)}$$

and we let χ_Ω to be the characteristic function of Ω. Furthermore, the total energy $E(t)$ is defined as

$$E(t) = \frac{1}{2}\left\{\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} + \|u_t(t,\cdot)\|^2_{L^2(\Omega)}\right\}. \tag{1.4}$$

Let $R > 0$ be an arbitrary real number so that $\partial\Omega \subset B_R(0) \equiv \{x \in \mathbf{R}^2; |x| < R\}$. Then, the local energy is defined as

$$E_{\Omega(R)}(t) = \frac{1}{2}\int_{\Omega(R)} \left\{|\nabla u(t,x)|^2 + |u_t(t,x)|^2\right\} dx, \tag{1.5}$$

where we set $\Omega(R) \equiv \Omega \cap B_R(0)$. We are concerned with a decay estimate of the local energy for a solution to (1.1), (1.2) and (1.3). We have some results on a decay of a local energy for the wave equations. For instance, we refer to [1], [4], [5], [7], [8], [9]. In particular, the results in [4], [5], [8] are much related to our interest in this paper.

First we show the unique existence of a weak solution in $C([0,\infty); H^1_0(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ to (1.1), (1.2) and (1.3) defined in the following. We can treat the higher dimension case that $\Omega \in \mathbf{R}^n$, $n \geq 2$. In this occasion, we shallproceed our argument based on the energy identity.

Theorem 1.1 *For each* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists a unique solution* $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ *to the problem* (1.1)*,* (1.2) *and* (1.3) *such that*

$$\frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2(\Omega)} = \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_1\|^2_{L^2(\Omega)}. \qquad (1.6)$$

For the proof see [2]

2. Main Theorem.

We now state our main theorem on the estimate in L^2 of a weak solution to (1.1), (1.2) and (1.3) in the two space dimension case, $\Omega \subset \mathbf{R}^2$. The key point is that we choose the initial data u_1 to be in the Hardy space, $\chi_\Omega u_1 \in \mathcal{H}^1(\mathbf{R}^2)$ (refer to [4], [5]).

First we give the definition of function spaces needed for our main theorem (refer to [3]).

Definition 2.1 (Hardy space) *The Hardy space consists of functions* f *in* $L^1(\mathbf{R}^n)$ *such that*

$$\|f\|_{\mathcal{H}^1(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \sup_{r>0} |\phi_r * f(x)| dx$$

is finite, where $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ *for* $r > 0$ *and* ϕ *is a smooth function on* $\mathbf{R}^n$ *with compact support in an unit ball with center of the origin* $B_1(0) = \{x \in \mathbf{R}^n; |x| < 1\}$.

We know that the definition dose not depend on choice of a function ϕ.

Definition 2.2 (functions of bounded mean oscillation) *Let* f *be a locally integrable in* $\mathbf{R}^n$*, denoted by* $f \in L^1_{loc}(\mathbf{R}^n)$*. We say that* f *is of bounded mean oscillation (abbreviated as BMO) if*

$$\|f\|_{BMO(\mathbf{R}^n)} = \sup_{B\subset\mathbf{R}^n} \frac{1}{|B|}\int_B |f(x) - (f)_B| dx < \infty,$$

where the supremum ranges over all finite ball $B \subset \mathbf{R}^n$*,* $|B|$ *is the n-dimensional Lebesgue measure of* B*, and* $(f)_B$ *denotes the mean value of* f *over* B*, namely* $(f)_B = \frac{1}{|B|}\int_B f(x)dx$.

The class of functions of BMO, modulo constants, is a Banach space with the norm $\|\cdot\|_{BMO}$ *defined above.*

Our main theorem is the following:

Theorem 2.3 *Suppose that the initial data* (u_0, u_1) *belongs to* $H_0^1(\Omega) \times L^2(\Omega)$ *and further satisfies* $\|\chi_\Omega u_1\|_{\mathcal{H}^1(\mathbf{R}^2)} < +\infty$*. Then, the solution* u *to the problem* (1.1)*,* (1.2) *and* (1.3) *satisfies*

$$\|u(t,\cdot)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} + C\|\chi_\Omega u_1\|^2_{\mathcal{H}^1(\mathbf{R}^2)} \qquad (2.1)$$

for all $t \ge 0$ *with a certain constant* $C > 0$.

We shall prepare the decisive Fefferman-Stein inequality, which means the duality between $\mathcal{H}^1(\mathbf{R}^n)$ and $BMO(\mathbf{R}^n)$, $(\mathcal{H}^1(\mathbf{R}^n))^* = BMO(\mathbf{R}^n)$. For the proof, see [3].

Theorem 2.4 (Fefferman-Stein inequality) *There is a positive constant C depending only on n such that if $f \in \mathcal{H}^1(\mathbf{R}^n)$ and $g \in BMO(\mathbf{R}^n)$, then*

$$\left|\int_{\mathbf{R}^n} f(x)g(x)dx\right| \le C\|f\|_{\mathcal{H}^1(\mathbf{R}^n)}\|g\|_{BMO(\mathbf{R}^n)}.$$

Theorem 2.5 *Assume that $\partial\Omega$ is star-shaped with respect to the origin. Let $R > 0$ be arbitrarily fixed such that $\partial\Omega \subset B_R(0)$. Then, for each $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with $\mathrm{supp}u_0 \cup \mathrm{supp}u_1 \subset \Omega(R)$ and further satisfies $\|\chi_\Omega u_1\|_{\mathcal{H}^1(\mathbf{R}^2)} < +\infty$, the weak solution $u(t, x)$ constructed in Theorem 1.1 to (1.1), (1.2) and (1.3) satisfies*

$$E_{\Omega(R)}(t) \le CE(0)(t-R)^{-1} \tag{2.2}$$

for all $t < R$, where the positive constant C depends only on the initial data (u_0, u_1).

As is well known, the finite propagation property of the wave equation implies that if the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ has a compact support, that is, $\mathrm{supp}u_0 \cup \mathrm{supp}u_1 \subset \Omega(R)$, then we have

$$\mathrm{supp}u(t,\cdot) \subset \Omega(R+t) \tag{2.3}$$

for each $t \ge 0$.

For the proof see [5]

3. Dissipative wave equation.

Our final result is concerned with the decay of solutions for the following dissipative wave equation :

$$\begin{aligned} u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) &= 0 \qquad \text{in} \quad (0,\infty)\times\Omega, &(3.1)\\ u(t,x) &= 0 \qquad \text{on} \quad (0,\infty)\times\partial\Omega, &(3.2)\\ u(t,x) = u_0, \quad u_t(t,x) &= u_1 \qquad \text{on} \quad \{t=0\}\times\Omega. &(3.3)\end{aligned}$$

To state the result, we need the well-posedness of the problem (3.1), (3.2) and (3.3).

Theorem 3.1 *For each $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique solution $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ to the problem (3.1), (3.2) and (3.3) such that*

$$\begin{aligned} &\frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t \|u_t(s,\cdot)\|^2_{L^2(\Omega)}ds \\ &= \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_1\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.4}$$

$$\frac{d}{dt}(u_t(t,\cdot), u(t,\cdot)) + \|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} + (u_t(t,\cdot), u(t,\cdot)) = \|u_t(t,\cdot)\|^2_{L^2(\Omega)}. \tag{3.5}$$

Theorem 3.2 *Suppose that the initial data* (u_0, u_1) *belongs to* $H_0^1(\Omega) \times L^2(\Omega)$ *and further satisfies* $\|\chi_\Omega(u_0 + u_1)\|_{\mathcal{H}^1(\mathbf{R}^2)} < +\infty$. *Then, the solution* u *to the problem* (3.1), (3.2) *and* (3.3) *satisfies*

$$(1+t)\|u(t,\cdot)\|_{L^2(\Omega)}^2 \leq C\{\|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 + \|\chi_\Omega(u_0+u_1)\|_{\mathcal{H}^1(\mathbf{R}^2)}^2\} \tag{3.6}$$

for all $t \geq 0$ *with a constant* $C > 0$ *independent of* $t \in [0, \infty)$.

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