Gierer-Meinhardt 系における点凝集現象と飽和効果

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In this talk, we are concerned with peak solutions (point-condensation solutions) to the following one-dimensional Gierer-Meinhardt system with saturation which is a model equation of the activator and the inhibitor in the biological pattern formation:

$$\text{(GM)} \qquad \begin{cases} 0 = \varepsilon^2 A'' - A + \frac{A^2}{H(1+\kappa A^2)},\ A > 0,\ x \in (-1,1), \\ 0 = DH'' - H + A^2,\ H > 0,\ x \in (-1,1), \\ A'(\pm 1) = H'(\pm 1) = 0, \end{cases}$$

where $\varepsilon, D > 0$, $\kappa \geq 0$. ε and D are the diffusion constants of the activator and the inhibitor, respectively. The saturation effect of the activator is given by the parameter κ. When $\kappa = 0$ (no saturation case), it is well-known that (GM) has a solution which concentrates at a finite number of points on $\overline{\Omega}$ for sufficiently small ε and large D. We call such a solution a peak solution. However, it is not trivial whether (GM) has a peak solution even if $\kappa > 0$. We will give a sufficient condition of κ for which point-condensation phenomena emerge.

We assume the following:
(A) $\kappa \geq 0$ depends on ε, and there exists a limit $\kappa\varepsilon^{-2} \to \kappa_0$ as $\varepsilon \to 0$ for some $\kappa_0 \in [0, \infty)$.

Theorem 1. *Fix $D > 0$ arbitrarily. Assume (A) and that the value $\kappa_0 \alpha_D^2$ is sufficiently small (α_D is defined below). Then, for sufficiently small ε, (GM) admits a 1-peak symmetric solution $(A_\varepsilon(x), H_\varepsilon(x))$ such that $A_\varepsilon(x)$ concentrates at $x = 0$. More precisely, there exists $\delta_\varepsilon \geq 0$ for each ε such that $\delta_\varepsilon \to \delta_0$ as $\varepsilon \to 0$ for certain $\delta_0 \geq 0$ which is decided by D and κ_0 and satisfies*

$$\delta_0 \Big(\int_{\mathbb{R}} w_{\delta_0}^2 \Big)^2 = \kappa_0 \alpha_D^2,\ \alpha_D := \frac{\sinh(2\theta)}{\theta \cosh^2 \theta},\ \theta := D^{-1/2},$$

A_ε takes the form

$$A_\varepsilon(x) = \frac{\alpha}{\varepsilon \int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \Big\{ w_{\delta_\varepsilon}\big(\frac{x}{\varepsilon}\big) + O(\varepsilon) \Big\},\ x \in (-1,1), \tag{1}$$

as $\varepsilon \to 0$, where the term $O(\varepsilon)$ is uniform in x, and w_δ is a unique solution to the following problem:

$$\begin{cases} w'' - w + \frac{w^2}{1+\delta w^2} = 0,\ w > 0,\ \text{in } \mathbb{R}, \\ w(0) = \max_{y \in \mathbb{R}} w(y),\ w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2}$$

Remark 1. *It is known that, there exists $\delta_* > 0$, and (2) has a unique symmetric solution w_δ for any $\delta \in [0, \delta_*)$. Moreover, w_δ has the following properties:*
(i) $w_\delta'(x) < 0$ for $x > 0$.
(ii) For some $C, c > 0$, it holds that

$$w_\delta(x), |w_\delta'(x)|, |w_\delta''(x)| \leq Ce^{-c|x|},\ x \in \mathbb{R}.$$

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