WAVE OPERATOR FOR THE SYSTEM OF THE DIRAC-KLEIN-GORDON EQUATIONS

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We study the system of the massive Dirac-Klein-Gordon equations in three space dimensions $x \in \mathbf{R}^3$

$$\left\{ \begin{array}{c} (\partial_t + \alpha \cdot \nabla + iM\beta)\,\psi = \lambda\phi\beta\psi, \\ \left(\partial_t^2 - \Delta + m^2\right)\phi = \mu\psi^*\beta\psi, \end{array} \right. \tag{0.1}$$

where the masses $m, M > 0$, the complex-valued vector-function

$$\psi = \psi\,(t,x) = (\psi_1\,(t,x)\,, \psi_2\,(t,x)\,, \psi_3\,(t,x)\,, \psi_4\,(t,x))^t$$

is a spinor field and a real-valued function $\phi = \phi\,(t,x)$ is a scalar field, $\lambda \in \mathbf{C}$, $\mu \in \mathbf{R}$,

$$\psi^* = \left(\overline{\psi_1}\,(t,x)\,, \overline{\psi_2}\,(t,x)\,, \overline{\psi_3}\,(t,x)\,, \overline{\psi_4}\,(t,x)\right)$$

denotes a transposed conjugate to the vector ψ. Here α_j and β are the 4×4 self-adjoint matrices with constant coefficients, such that $\beta^2 = \alpha_j^2 = I$, $\alpha_j\beta + \beta\alpha_j = O$, $\alpha_j\alpha_k + \alpha_k\alpha_j = O$, for $j, k = 1, 2, 3$, $j \neq k$, $I = [\delta_{jk}]_{1\le j,k\le 4}$, with the Kronecker symbols $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$.

Our purpose is to show the existence of the wave operator and to improve the regularity conditions on the data supposed in paper [12]. The initial value problem for the massive Dirac-Klein-Gordon equations (0.1) was studied by reducing it to the system of the nonlinear Klein-Gordon equations. Denote $\mathcal{D}_{\sigma,M} = \partial_t + \sigma\,(\alpha \cdot \nabla + iM\beta)$. In view of the properties of the matrices α_j, β we have

$$\mathcal{D}_{-1,M}\mathcal{D}_{+1,M}\psi = \left\{\partial_t^2 - (\alpha \cdot \nabla + iM\beta)\,(\alpha \cdot \nabla + iM\beta)\right\}\psi = \left(\partial_t^2 + \langle i\nabla\rangle_M^2\right)\psi,$$

where $\langle i\nabla\rangle_M^2 = M^2 - \Delta$. Hence multiplying both sides of the Dirac equation $\mathcal{D}_{+1,M}\psi = \phi\beta\psi$ by $\mathcal{D}_{-1,M}$, we obtain

$$\begin{aligned} \left(\partial_t^2 + \langle i\nabla\rangle_M^2\right)\psi &= \lambda\mathcal{D}_{-1,M}\phi\beta\psi \\ &= \lambda\left((\mathcal{D}_{-1,M}\phi)\,\beta\psi - iM\phi I\psi + \phi\beta\mathcal{D}_{+1,M}\psi\right) \\ &= \lambda\left((\mathcal{D}_{-1,M}\phi)\,\beta - iM\phi I + \lambda\phi^2 I\right)\psi. \end{aligned}$$

Thus from the Dirac-Klein-Gordon equation (0.1) it follows a system of nonlinear Klein-Gordon equations

$$\left\{ \begin{array}{c} \left(\partial_t^2 + \langle i\nabla\rangle_M^2\right)\psi = \lambda\left((\mathcal{D}_{-1,M}\phi)\,\beta - iM\phi I + \lambda\phi^2 I\right)\psi, \\ \left(\partial_t^2 + \langle i\nabla\rangle_m^2\right)\phi = \mu\psi^*\beta\psi, \end{array} \right. \tag{0.2}$$

where $\langle i\nabla\rangle_m^2 = m^2 - \Delta$. However, the existence of the wave operators for the massive Dirac-Klein-Gordon equations (0.1) is not clear, since the solutions of system (0.2)

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are constructed by the ten given final data, moreover the four of them correspond to the solution ψ_1 of the homogeneous time-inverse Dirac equation $\mathcal{D}_{-1,M}\psi_1 = 0$.

To state the problem of existence of the wave operator for system (0.1) we define the free Dirac evolution group

$$\mathcal{V}_{D,M}(t) = I\cos\left(t\langle i\nabla\rangle_M\right) - (\alpha\cdot\nabla + iM\beta)\langle i\nabla\rangle_M^{-1}\sin\left(t\langle i\nabla\rangle_M\right),$$

and the free Klein-Gordon evolution group

$$\mathcal{V}_{KG,m}(t) = \begin{pmatrix} \cos\left(\langle i\nabla\rangle_m t\right) & \sin\left(\langle i\nabla\rangle_m t\right) \\ -\sin\left(\langle i\nabla\rangle_m t\right) & \cos\left(\langle i\nabla\rangle_m t\right) \end{pmatrix}.$$

Then we look for the solutions of system (0.1) which obey the following final state conditions

$$\lim_{t\to\infty}\left\|\mathcal{V}_{D,M}(-t)\psi(t) - \psi^+\right\|_{\mathbf{X}_1} = 0$$

and

$$\lim_{t\to\infty}\left\|\mathcal{V}_{KG,m}(-t)\begin{pmatrix}\phi(t) \\ \langle i\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix} - \begin{pmatrix}\phi_1^+ \\ \langle i\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{\mathbf{X}_2} = 0$$

for the six given final data $\psi^+ \in \mathbf{X}_1$, $\left(\phi_1^+, \phi_2^+\right) \in \mathbf{X}_2$ with some Hilbert spaces $\mathbf{X}_1$ and $\mathbf{X}_2$ which are defined explicitly later.

The problem of the existence of the wave operator can be formulated then in the form of the integral equations

$$\mathcal{V}_{D,M}(-t)\psi(t) = \psi^+ - \int_t^\infty \mathcal{V}_{D,M}(-s)\phi\beta\psi(s)\,ds \tag{0.3}$$

and

$$\begin{aligned}&\mathcal{V}_{KG,m}(-t)\begin{pmatrix}\phi(t) \\ \langle i\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix} \\ &= \begin{pmatrix}\phi_1^+ \\ \langle i\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix} - \int_t^\infty \mathcal{V}_{KG,m}(-s)\begin{pmatrix}0 \\ \langle i\nabla\rangle_m^{-1}\psi^*\beta\psi(s)\end{pmatrix}ds\end{aligned} \tag{0.4}$$

for the given final data $\psi^+ \in \mathbf{X}_1$, $\left(\phi_1^+, \phi_2^+\right) \in \mathbf{X}_2$. If there exists a unique solution $\left(\psi(t), \phi(t), \langle i\nabla\rangle_m^{-1}\partial_t\phi(t)\right)$ of system (0.3)-(0.4) for the given final data $\left(\psi^+, \phi_1^+, \phi_2^+\right)$, then the wave operator $\mathcal{W}^+$ for the system (0.3)-(0.4) denotes the mapping

$$\begin{pmatrix}\psi(t) \\ \phi(t) \\ \langle i\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix} = \mathcal{W}^+\begin{pmatrix}\psi^+ \\ \phi_1^+ \\ \langle i\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}.$$

We introduce some notations. Denote the usual Lebesgue space by $\mathbf{L}^p$. Weighted Sobolev space

$$\mathbf{H}_p^{m,k} = \left\{\phi : \|\phi\|_{\mathbf{H}_p^{m,k}} \equiv \left\|\langle x\rangle^k\langle i\nabla\rangle^m\phi\right\|_{\mathbf{L}^p} < \infty\right\},$$

where $m, k \in \mathbf{R}$, $1 \le p \le \infty$. We also write for simplicity $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$, $\mathbf{H}^m = \mathbf{H}_2^{m,0}$, $\mathbf{H}_p^m = \mathbf{H}_p^{m,0}$, so we usually omit the index 0 if it does not cause a confusion. We use the same notations for the vector-functions.

If we can show that $D(\mathcal{W}^+) = R(\mathcal{W}^+)$, where $D(\mathcal{W}^+)$ is the domain and $R(\mathcal{W}^+)$ is the range of the wave operator $\mathcal{W}^+$, then we can easily construct the scattering operator. The existence of solutions for the cubic nonlinear Klein-Gordon equations in the low energy space along with the property $D(\mathcal{W}^+) = R(\mathcal{W}^+)$ were obtained in [11] by using the $\mathbf{L}^p - \mathbf{L}^q$ method and the Strichartz type estimates. The cubic nonlinear Dirac equation $(\partial_t + \alpha \cdot \nabla + iM\beta)\psi = \lambda(\psi^* \beta \psi)\psi$ was studied in paper [10], where the scattering operator was obtained in $\mathbf{H}^s$ with $s > 1$.

We now survey some works concerning system (0.2). The existence of the global small solutions to the Cauchy problem for the quadratic nonlinear Klein-Gordon equations including (0.2) was shown in [8] by applying the time decay estimates through the operators $(\partial_j, \partial_t, x_j\partial_t + t\partial_j)_{1 \le j \le 3}$ and using the hyperbolic coordinates. The use of the hyperbolic coordinates implies the consideration of the problem inside of the light cone and so yields the compactness condition on the initial data. In papers [1], [2], [3], [7] the method of [8] was improved and the compactness condition on the data was removed however the higher order Sobolev spaces for the initial data were implemented. The global in time existence of small solutions to the Cauchy problem for a system of quadratic nonlinear Klein-Gordon equations including (0.2) was shown in [6] for the case of small initial data $(\psi(0), \partial_t\psi(0), \phi(0), \partial_t\phi(0))$ in the space $\left(\mathbf{H}^{4,3} \times \mathbf{H}^{3,3}\right)^5$, moreover the inverse wave operator was constructed from the neighborhood of the origin in the space $\left(\mathbf{H}^{4,3} \times \mathbf{H}^{3,3}\right)^5$ to the neighborhood at the origin in the space $\left(\mathbf{H}^{4,1} \times \mathbf{H}^{3,1}\right)^5$. In paper [12], it was shown the existence of the scattering operators for system (0.2) from the neighborhood of the origin in the space $\left(\mathbf{H}^{5/2,1} \times \mathbf{H}^{3/2,1}\right)^4 \times \left(\mathbf{H}^{3,1} \times \mathbf{H}^{2,1}\right)$ to the neighborhood of the origin in the same space.

Our main result is the following. Denote $\frac{90}{37} < q < 6$ and $\mu = \frac{5}{4} - \frac{5}{2q}$. Note that we can choose $\mu = \frac{1}{4}$ when we take $q = \frac{5}{2}$.

Theorem 0.1. *Let the final data* $\psi^+ \in \left(\mathbf{H}^{\frac{3}{2}+\mu,1}\right)^4$, $\left(\phi_1^+, \phi_2^+\right) \in \mathbf{H}^{2+\mu,1} \times \mathbf{H}^{1+\mu,1}$. *Then there exists* $\varepsilon > 0$ *such that for any final data* $\left(\psi^+, \phi_1^+, \phi_2^+\right)$ *satisfying estimate*

$$\left\|\psi^+\right\|_{\mathbf{H}^{\frac{3}{2}+\mu,1}} + \left\|\phi_1^+\right\|_{\mathbf{H}^{2+\mu,1}} + \left\|\phi_2^+\right\|_{\mathbf{H}^{1+\mu,1}} \le \varepsilon,$$

there exists a unique global solution

$$\begin{aligned} &\left(\mathcal{V}_{D,M}(-t)\psi(t), \mathcal{V}_{KG,m}(-t)\begin{pmatrix} \phi(t) \\ \langle i\nabla\rangle_m^{-1}\partial_t\phi(t) \end{pmatrix}\right) \\ \in\ &\left(\mathbf{C}\left([0,\infty); \mathbf{H}^{\frac{3}{2}+\mu,1}\right)\right)^4 \times \begin{pmatrix} \mathbf{C}\left([0,\infty); \mathbf{H}^{2+\mu,1}\right) \\ \mathbf{C}\left([0,\infty); \mathbf{H}^{2+\mu,1}\right) \end{pmatrix} \end{aligned}$$

for system (0.1) with the final state condition

$$\begin{aligned} &\left\|\mathcal{V}_{D,M}(-t)\psi(t) - \psi^+\right\|_{\mathbf{H}^{\frac{3}{2}+\mu,1}} \\ &+ \left\|\mathcal{V}_{KG,m}(-t)\begin{pmatrix} \phi(t) \\ \langle i\nabla\rangle_m^{-1}\partial_t\phi(t) \end{pmatrix} - \begin{pmatrix} \phi_1^+ \\ \langle i\nabla\rangle_m^{-1}\phi_2^+ \end{pmatrix}\right\|_{\mathbf{H}^{2+\mu,1}} \to 0 \end{aligned} \tag{0.5}$$

as $t \to \infty$. *Moreover the estimate is true*

$$\left\|\mathcal{V}_{D,M}(-t)\psi(t)\right\|_{\mathbf{H}^{\frac{3}{2}+\mu,1}} + \left\|\mathcal{V}_{KG,m}(-t)\begin{pmatrix} \phi(t) \\ \langle i\nabla\rangle_m^{-1}\partial_t\phi(t) \end{pmatrix}\right\|_{\mathbf{H}^{2+\mu,1}} \le C\varepsilon$$

for all $t \ge 0$.

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